Jean-Michel Bony

The mathematical work of Louis Boutet de Monvel

Memorial conference in honor of Louis Boutet de Monvel
École Normale Supérieure, June 20–24, 2016

Séminaire CARTAN-SCHWARTZ. 16. 1963/64

2e édition corrigée

THÉORÈME D'ATIYAH-SINGER
SUR L'INDICE D'UN OPÉRATEUR DIFFÉRENTIEL ELLIPTIQUE

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TABLE DES MATIÈRES

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Boundary problems for pseudo-differential operators.

2× *J. Anal. Math.*+*Ann. Inst. Fourier*+***Acta Math. 1971***

$$A = \begin{pmatrix} P+G & K \\ T & Q \end{pmatrix} \quad : \quad \begin{matrix} C^\infty(\overline{\Omega}, E) \\ \bigoplus \\ C^\infty(\partial\Omega, F) \end{matrix} \quad \longrightarrow \quad \begin{matrix} C^\infty(\overline{\Omega}, E') \\ \bigoplus \\ C^\infty(\partial\Omega, F') \end{matrix}$$

E, E' : vector bundles on $\overline{\Omega}$; F, F' on $\partial\Omega$.

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The transmission property. Characterization of P s. t.

$$f \longmapsto \left(P \boxed{\tilde{f}}\right)\Big|_{\overline{\Omega}} \quad \text{maps } C^\infty(\overline{\Omega}) \text{ into } C^\infty(\overline{\Omega}) \ .$$

extension by 0 outside Ω

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with p_k homogeneous in ξ of degree $d_k \searrow -\infty$,

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OK for differential operators and parametrices.

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Poisson operators $K\colon C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$

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Poisson operators K: $C^\infty(\partial\Omega) \to C^\infty(\overline{\Omega})$

Trace operators T: $C^\infty(\overline{\Omega}) \to C^\infty(\partial\Omega)$

— usual ones (restriction of derivatives)

— but also adjoints of Poisson operators.

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ΨDO on $\partial\Omega$. $Q\colon C^\infty(\partial\Omega) \to C^\infty(\partial\Omega)$

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Singular Green operators G: $C^\infty(\overline{\Omega}) \to C^\infty(\overline{\Omega})$

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infinitely regularizing inside Ω

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General G can be expanded as $\sum K_j T_j$.

Symbolic calculus. For $A = \begin{pmatrix} P+G & K \\ T & Q \end{pmatrix}$

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$\{h \in C^\infty(\mathbb{R})\}$ s.t. analytic extension in the lower half-plane, regular pole and $\to 0$ at infinity.

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Full symbol: $k(x', \xi', \xi_n) \in C^\infty(\mathbb{R}^{n-1} \times \mathbb{R}^n)$

depending on ξ_n as an element of H^+

depending on (x', ξ') as a classical symbol

$$Kf(x) = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \widehat{f}(\xi')\, d\xi' \int e^{ix_n \xi_n} k(x', \xi', \xi_n)\, d\xi_n.$$

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The index formula for elliptic A

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One can conjugate : $B_1 \circ A \circ B_2 = \begin{pmatrix} \widetilde{P} & 0 \\ 0 & \widetilde{Q} \end{pmatrix}$

elliptic of index 0

$\widetilde{Q}$ elliptic on $\partial\Omega$; $\widetilde{P}$ elliptic and $=$ Id near $\partial\Omega$

Analytic pseudodifferential calculus (also Gevrey)

Ann. Fourier 67 (with P. Krée) + Ann. Fourier 69

Symbol p = formal sum $\sum p_k(x, \xi)$ (estimates...)

$\nwarrow$ hom. in ξ degree $m-k$

$p \rightsquigarrow P$ ΨDO unique up to a regularizing one
(i.e. analytic functionals $\to$ analytic functions)

P maps analytic functionals into hyperfunctions,
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Formal norm of p

$$N_\sigma(p,T) = \sum_{\alpha,\beta,k} \left(\frac{2(2n)^{-k} k!}{(k+|\alpha|)!^\sigma (k+|\beta|)!} \right) \left| \partial_x^\alpha \partial_\xi^\beta p_k \right| T^{2k+|\alpha+\beta|}.$$

$$N(p\circ q, T) \ll N(p,T)\cdot N(q,T)$$

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LMP 2008 : Formal norms and star-exponentials.

Operators with double characteristics

Invent. 74 & CPAM 74 (w. F. Treves) + ***CPAM 74*** *+ Asterisque 76 (w. A. Grigis and B. Helffer).*

Given : a ΨDO P of order m with symbol $p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \dots$ s. t. p_m vanishes exactly at order 2 on $\Sigma \subset T^*\mathbb{R}^n$.

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Case Σ symplectic of codimension 2ν. For $(x,\xi) \in \Sigma$

Transversal Hessian $\rightarrow$ operator on $N^*\Sigma$ (via symplectic form) $\rightarrow$ eigenvalues $\pm i\lambda_j$, $j = 1, \ldots, \nu$.

Invariant $I_2(P) = p_{m-1} - \frac{1}{2i} \sum \frac{\partial^2 p_m}{\partial \xi_k \partial x_k} + \sum_1^\nu \lambda_j$

TH: P is hypoelliptic with loss of 1 derivative and has a parametrix $\iff$ $\forall \alpha \in \mathbb{N}^\nu, \ \sum_j \alpha_j \lambda_j + I_2(P) \neq 0.$

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A first symbolic calculus. In local coordinates:

$x = (y,t) \in \mathbb{R}^{n-\nu} \times \mathbb{R}^\nu$; Σ defined by $t = \tau = 0$

$p \in S^{m,k} \Longleftrightarrow |p(x,\xi)| \leq c\,|\xi|^m\, d_\Sigma^k, \quad d_\Sigma = \Big(t^2 + \dfrac{\tau^2}{|\xi|^2} + \dfrac{1}{|\xi|}\Big)^{1/2}$

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Good symb. calc. for Op($S^{m,k}$), composition with FIO.

$P \in$ Op($S^{m,2}$)

A first approximate inverse : $\exists Q_1 \in \mathrm{Op}(S^{-m,-2})$ s.t.

$$PQ_1 = I + R_1, \quad R_1 \in \mathrm{Op}\,\mathcal{H}^0, \quad \mathcal{H}^0 = \cap\, S^{-N/2,-N}$$

fast decay w.r.t. $(|\xi|\, t^2 + \tau^2/\, |\xi|)$, regularize only outside Σ.

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Hermite operators

$$f(y) \mapsto Hf(y,t) = (2\pi)^{n-\nu} \int e^{iy\cdot\eta} h(y,t,\eta) \hat{f}(\eta)\, d\eta$$

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Basic Hermite op. H_α of symbol $|\eta|^{\nu/4}\, h_\alpha(|\eta|^{1/2}\, t)$

h_α : classical Hermite functions of ν variables.

$$\forall R \in \mathrm{Op}\,\mathcal{H}^0, \quad R = \sum_{\alpha,\beta} H_\alpha R_{\alpha\beta} H_\beta^\star$$

with $R_{\alpha\beta}$ rapidly decreasing sequence of ΨDO on Y.

Computation: $PH_\alpha = \sum H_\beta Q_{\alpha\beta} +$ l.o.t.

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Remember $PQ_1 = I + R_1, \quad R_1 \in \mathcal{H}^0$

Inverting the matrix solves $PQ_2 = R_1 + R_2, \quad R_2 \in \mathcal{H}^{-1/2}$,

$P(Q_1 - Q_2) = I +$ regularizing

$\rightarrow$ bilateral parametrix $\rightarrow$ hypoellipticity

two remarks

Boundary Cauchy-Riemann. X real hypersurface of $\mathbb{C}^n$,

The $\overline{\partial}_b$ complex on $(0,\star)$-forms

The Kohn Laplacian $\Box_b = \overline{\partial}_b \overline{\partial}_b^\star + \overline{\partial}_b^\star \overline{\partial}_b$

$\Sigma = \mathrm{char}(\Box_b)$ symplectic $\iff$ Levi form non degenerate

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If the Levi form has signature $(q, n-1-q)$ at $(x, \xi) \in \Sigma$

Then $\Box_b$ is microlocally hypoelliptic and $\overline{\partial}_b$ has no cohomology except in dimension q.

Global embeddability of abstract CR-manifold

Data: X real manifold, dimension $2n-1$, and a subbundle $T'' \subset \mathbb{C}TX$, dim. $n-1$ which mimics $\overline{\partial}_b$

formally integrable, Levi form.

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TH. If X is compact, if $2n-1 > 3$ and the Levi form is positive def. then $\exists\, Y$, complex manifold with boundary $\partial Y = X$ s.t. $T'' =$ antiholomorphic vectors tangent to X.

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No cohomology in degree 1, except for $2n-1=3$

Bergman and Szegö kernels

Asterisque 76 (with J. Sjöstrand).

$\Omega \subset \mathbb{C}^n$ strictly pseudoconvex ($\{z \,|\, \rho(z) > 0\}$)

Bergman projector in $L^2(\Omega)$ on holomorphic functions

Szegö projector in $L^2(\partial\Omega)$ on their traces ($\ker \overline{\partial}_b$)

Phase function $\Psi \in C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ such that

$$\Psi(x,x) = \frac{1}{i}\rho(x), \quad \Psi(x,y) = -\overline{\Psi(y,x)},$$

$\overline{\partial}_x \Psi$ and $\partial_y \Psi$ vanishes of infinite order for $y = x$.

TH. $\exists\, F, G \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ s.t.

$$B = F(-i\Psi)^{-n-1} + G \log(-i\Psi)$$

$$B(x,y) = \int_0^\infty e^{it\Psi(x,y)} b(x,y,t)\, dt \pmod{C^\infty}$$

$$b(x,y,t) \sim \sum t^{n-k} b_k(x,y) \text{ symbol of order } n$$

Analogous results for $S(x,y)$ on $\partial\Omega \times \partial\Omega$.

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Analogous results for $S(x,y)$ on $\partial\Omega\times\partial\Omega$.

Microlocal model of Szegö projector $L^2(\mathbb{R}^p_x) \to L^2(\mathbb{R}^{p+q}_{x,y})$

$$H_0 f(x,y) = (2\pi)^{-p}\int_{\mathbb{R}^p} e^{ix\cdot\xi - |y|^2|\xi|/2}\Big(\frac{|\xi|}{\pi}\Big)^{q/4}\hat{f}(\xi)\,d\xi.$$

Isometry $L^2(\mathbb{R}^p) \leftrightarrow L^2(\mathbb{R}^{p+q}) \cap \mathcal{H}$, $\quad \mathcal{H} = \ker\left\{\partial_{y_j} + y_j\,|D_x|\right\}$

$S_0 = H_0^\star H_0 =$ projector in $L^2(\mathbb{R}^p)$ on traces of $\mathcal{H}$.

S and S_0 can be conjugated microlocally by FIO.

$p = n,\ q = n-1$

The logarithmic term. $B = F(-i\Psi)^{-n-1} + G \log(-i\Psi)$

Not the dominant term, but a local biholomorphic invariant.

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TH. (1988) For $n = 2$, if G vanishes at order 2 on $\partial\Omega$ near $z_0 \in \partial\Omega$ then $\partial\Omega$ is locally biholomorphic to a portion of sphere.

Vanishing at order 1 is not sufficient.

Toeplitz operators

Inventiones 79 + Princeton Univ. Pr. 81 (w. V. Guillemin)

Ω strictly pseudoconvex, $X = \partial\Omega$

Σ^+ = the half-line bundle over X (a conic symplectic manifold) = points of T^*X, characteristic for $\overline{\partial}_b$ and where $\overline{\partial}_b$ is not hypoelliptic.

DEF. Toeplitz operator of degree m: operator $T_Q = S\,Q$

where Q is a ΨDO of degree m on X. Szegö projector

T_Q maps $H^s \cap \ker \overline{\partial}_b$ into $H^{s-m} \cap \ker \overline{\partial}_b$.

Main results

Toeplitz operators form an algebra of pseudolocal operators, isomorphic to ΨDO in n real variables.

Principal symbol of $T_Q =$ restriction to Σ^+ of $\sigma_m(Q)$

Symbolic calculus OK

Elliptic ones (invertible symbol) have a parametrix and thus a finite index in spaces $H^s \cap \ker \overline{\partial}_b$

Index formula for elliptic Toeplitz.

Quantization of conic symplectic manifolds

X compact mfold, $\Sigma \subset T^*X$ closed conic symplectic.

Toeplitz structure: a projector π_Σ in $L^2(X)$ microlocally equivalent (via FIO) to the model of Szegö projector.

$H_\Sigma =$ image of π_Σ

Toeplitz operator: $T_Q = \pi_\Sigma Q$ maps $H_\Sigma \cap C^\infty$ into itself.

Principal symbol: restriction to Σ of that of Q.

Everything extends to this situation.

Spectral theory

T : selfadjoint Toeplitz operator of order 1, $\sigma(T) > 0$.

Spectrum: $\lambda_1 \leq \lambda_2 \leq \lambda_3 \ldots \to +\infty$.

Generating function: $E(t) = \mathrm{Tr}(e^{itT}) = \sum e^{i\lambda_j t}$.

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Trace formula $E(t) \cong \sum_{\gamma} C_\gamma (t - \tau_\gamma + i0)^{-1} \pmod{L^1_{\mathrm{loc}}(\mathbb{R}^+)}$,

periodic bicharacteristics period

primitive period

$$C_\gamma = \frac{\tau_\gamma^{\#}}{|I - P_\gamma|^{1/2}} \exp\left(i \int_\gamma \sigma_{\mathrm{sub}}\right)$$

Poincaré map (non deg.)

Consequences

- Weyl law:

$$\#\left\{j \,\middle|\, \lambda_j < \lambda\right\} = \frac{\mathrm{vol}\,\{p \in \Sigma \mid \sigma_T(p) \leq 1\}}{(2\pi)^\nu}\,\lambda^\nu + O(\lambda^{\nu-1}),$$

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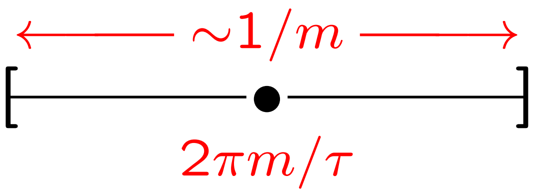
- Concentration of eigenvalues in intervals I_m (cf. Colin de Verdière for ΨDO)

$$\#\left\{j \,\middle|\, \lambda_j \in I_m\right\} = \text{Pol}(m) \text{ for } m \text{ large}$$



Assuming same period τ and same integral of subprincipal symbol

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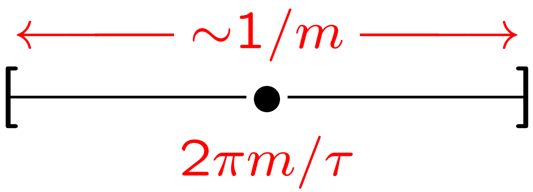

Assuming same period τ and same integral of subprincipal symbol

- Definition of a Hilbert polynomial for a compact symplectic manifold (assuming $[\omega] \in H^2(M, \mathbb{Z})$)

Relative index formula

Astérisque 85 + Ann. Sci. ENS (with B. Malgrange)

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Given: a morphism f of analytic varieties

a coherent $\mathcal{D}_Y$-module P, relatively elliptic

a good filtration of P and $Z \supset \mathrm{char}(P)$

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image $f_+(P)$ (M. Kashiwara) in der. cat. of $\mathcal{D}_X$-modules

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Real analogue for almost elliptic systems

Asymptotic equivariant index of Toeplitz operators

Progr. Math. 2011 (w. E. Leichtnam, X. Tang, A. Weinstein)

- Equivariant Toeplitz calculus.
- New proof of the Atiyah-Weinstein conjecture on the index of FIO.
- Relative index of CR structure.

Mathematical physics

Seminar *Mathematics and Physics* 1979–1982

with A. Douady and J.-L. Verdier

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$H_0 = -\Delta + \sum \lambda_j x_j^2$; $H = H_0 + V$; $V(x) \leq C\,|x|^\gamma, \gamma < 1, \ldots$

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- e^{itH} and e^{itH_0} have the same wave front
- $\mathrm{Tr}(e^{itH})$ has its singularities on the periods of H_0

$e^{-itH_0}e^{itH}$ not a ΨDO but . . .

Deformation Quantization

Math. Phys. Anal. Geom. 99 + Lett. Math. Phys. 2009

- Extension of the notion of star-product

to complex Poisson manifolds

Case of pseudodifferential or Toeplitz algebra.

- Classification of star-algebras, diff-algebras and

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- star-products of a given degree of homogeneity k

$k \geq 3 \to$ trivial ; $k = 2 \to$ Moyal product;

$k = 1 \to$ a Lie algebra structure on the dual.

Residual trace of Toeplitz or pseudodifferential projectors

L. Boutet de Monvel

Residual trace of Toeplitz or pseudodifferential projectors

Colloque en honneur de Gilles Lebeau, Nice, juin 2014

UPMC, F75005, Paris, France - louis.boutet-de-monvel@orange.fr

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"noncommutative residue" of Wodzicki

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Vanishing theorem

Let $\mathcal{E}$ denote the Toeplitz algebra (mod C^∞) of a contact manifold, or the ψDOalgebra (which is a special case when the base manifold is a cotangent sphere).[4]
Our main result is

Theorem

If P is any projector with coefficients in $\mathcal{E}$, of degree 0, then its residual trace is 0.

In other words the residual index of its range vanishes.

[4] recall [4],[10], that the Toeplitz algebra mod C^∞ is uniquely defined, up to non unique isomorphism.

When to the sessions of sweet silent thought
I summon up remembrance of things past,
I sigh the lack of many a thing I sought,
And with old woes new wail my dear time's waste:
Then can I drown an eye, unused to flow,
For precious friends hid in death's dateless night

W. Shakespeare