Spectral asymptotics for sub-Riemannian Laplacians
(work in progress with Luc Hillairet and Emmanuel Trélat)

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Memorial conference in honor of Louis Boutet de Monvel
ENS, Paris, june 2016

Crossing our paths with Louis:

Paris 7 (71-75): Louis advisor for my “second thesis” (also called “proposition donnée par la faculté”)

Grenoble (77-79): colleagues at Institut Fourier

American Academy of Arts and Sciences (2012): Louis elected

Collaboration on a paper (2012): `arXiv:1209.5165 [math-ph]`

The action of pseudo-differential operators on functions harmonic outside a smooth hyper-surface [Louis Boutet De Monvel (IMJ), Yves Colin De Verdière (IF) (Submitted on 24 Sep 2012)]

We consider a smooth hyper-surface Z of a closed Riemannian manifold X. Let P be the Poisson operator associating to a smooth function on Z its harmonic extension on $X \setminus Z$. If A is a pseudo-differential operator on X of degree < 3, we prove that $B = P^*AP$ is a pseudo-differential operator on Z and calculate the principal symbol of B.

Le 28 juin 2012

Boutet de Monvel Louis <louis.boutet-de-monvel@orange.fr> a écrit:

Cher Yves,

merci de tes mels. Je suis loin d'être le principal auteur, mais je serais quand même très content de cosigner cette note - comme ça, ça fera au moins un papier cosigné, en 35 ans ! Je prends quand même un ou deux jours de plus pour relire la deuxième version.
Amitiés, Louis

Sub-Riemannian (sR) Laplacians are self-adjoint hypo-elliptic operators "à la Hörmander" on which Louis contributed a lot. Works in the seventies and the eighties by many people.

One of our main objectives while starting this work was to get *Quantum Ergodicity* results for sR Laplacians and more generally to understand the link between their spectral asymptotics and some classical dynamics *(not always the geodesic flow).*

We succeeded for *3D contact distributions:* the relevant dynamics is then the associated *Reeb vector field.*

I will also present some results for some generic singularities of the distribution (*Grushin* and *Martinet* singularities). As far as we know, Weyl asymptotics were not known in these cases. We show that most eigenfunctions do concentrate on the singular set.

Topics.–

- Sub-Riemannian Laplacians
- Weyl measures
- Quantum limits and Quantum ergodicity
- The 3D contact case
- Singular cases: Grushin and Martinet.

Sub-Riemannian Laplacians.–

sR Laplacians are locally given by

$$\Delta = \sum_{m=1}^{l} X_m^\star X_m$$

where the family of vector fields (X_m) satisfies the Hörmander bracket generating condition: the iterated brackets of the vector fields (X_m) generate the tangent space.

A more global and intrinsic definition is

- X a smooth compact connected manifold of dimension d with a smooth measure μ

- A smooth linear bundle map j from a vector bundle E on X into TX satisfying the bracket generating condition (the iterated brackets of images of sections of E generate TX)

- A smooth metric g on E.

To these data is associated a finite distance using the lengths of "horizontal" paths, ie tangent to $j(E)$.

We do not assume that E is a sub-bundle of TX (see Grushin case).

The distribution is called *equi-regular* if E is a sub-bundle of TX and, defining $E = E_1 \subset E_2 \subset \cdots E_r = TX$ with $E_r = E_{r-1} + [E, E_{r-1}]$, each E_j is a sub-bundle of TX. In this case $Q_0 = \dim E_1 + 2 \dim(E_2/E_1) + \cdots$ is the Hausdorff dimension of X for the sR distance.

We define the sR Laplacian $\Delta_{g,\mu}$ as the self-adjoint operator on $L^2(X,\mu)$ which is the Friedrichs extension of the closure of the quadratic form $D(f) := \int_X \|df\|_g^2 d\mu$ where $\|df\|_g$ is the $g-$norm of $d(f \circ j)$.

Locally, if $(e_1, e_2, \cdots, e_l)$ is a local orthonormal frame for (E, g),

$$\Delta_{g,\mu} = X_1^\star X_1 + X_2^\star X_2 + \cdots$$

where the vector fields X_m's are the images of the e_m's by j and the adjoints are taken w.r. to μ.

The principal symbol of $\Delta_{g,\mu}$ is the co-metric $g^\star$ defined by $g^\star(x,\xi) = \|\xi \circ j\|_{g_x}^2$. The sR Laplacians are not elliptic at the points where j_x is not surjective. The characteristic manifold $\Sigma \subset T^\star X$ is the orthogonal of $j(E)$. The sub-principal symbol vanishes and different choices of μ give operators unitarily equivalent up to a bounded operator. The main spectral asymptotics depends only of the metric.

It follows from Hörmander's Theorem that $\Delta_{g,\mu}$ is sub-elliptic and has a compact resolvent, hence a discrete spectrum and a spectral decomposition (ϕ_n, λ_n). We are interested in spectral asymptotics for sR Laplacians $\Delta_{g,\mu}$.

Link with magnetic fields.–

(M, h) is a closed Riemannian manifold and $X \to M$ is a S^1-bundle with a connection ∇. The sR distribution E is the horizontal space of ∇ and g is the pull-back of h on E. The curvature $B \in \Omega^2(M)$ of ∇ is the magnetic field.

The sR laplacian on X commutes with the S^1 action. Using Fourier expansion allows a decomposition $\Delta_{sR} = \oplus_{n\in\mathbb{Z}}\Delta_n$ where Δ_n is a Schrödinger operator on M with magnetic field nB.

If M is of dimension 2, transverse vanishing of B along a curve Y corresponds to a Martinet singularity in X. This was first observed by Richard Montgomery and leads him to the discovery of existence of singular geodesics, i.e. geodesics which are not projections of integral curves of the Hamiltonian vector field.

If M is of dimension 3 and B is non vanishing, the corresponding sR structure is equi-regular in dimension 4 and is called quasi-contact. Magnetic lines are projections of the singular geodesics. QE in this case could possibly come from the ergodicity of the magnetic vector field.

Weyl measures.–

If Δ is an operator with compact resolvent and $N(\lambda) := \#\{\lambda_n \leq \lambda\}$ the spectral counting function, we define (if they exist!) the Weyl (probability) measures as follows:

The local Weyl measure dw_Δ on X

$$\int_X f dw_\Delta = \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n\leq\lambda} \int_X f|\phi_n|^2 d\mu$$

The micro-local Weyl measure dW_Δ on the co-sphere bundle $S^\star X$

$$\int_{S^\star X} a dW_\Delta = \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n\leq\lambda} \langle \mathrm{Op}(a)\phi_n|\phi_n\rangle$$

where $\mathrm{Op}(a)$ is any ΨDO of degree 0 with principal symbol a.

If $p : S^\star X \to X$ is the projection, $p_\star dW_\Delta = dw_\Delta$.

The microlocal Weyl measure is related to the average *correlation* of the eigenfunctions: if, locally,

$$C(x,y) = \lim_{\lambda \to \infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} \phi_n(x + y/2)\bar{\phi}_n(x - y/2),$$

using Weyl quantization, one gets that the correlation is the Fourier transform w.r. to ξ of the microlocal Weyl density:

$$C(x,y) = \int_{\mathbb{R}^d} e^{-i\xi y} \frac{dW_{\Delta}(x,\xi)}{dx} \ .$$

General facts:

- The measures dw_{Δ} and dW_{Δ} depend only of (E, j, g) and not of μ. dW_{Δ} is invariant by the involution $(x, \xi) \to (x, -\xi)$.

- In the equi-regular case, dw_{Δ} is smooth and is in general distinct from the normalized Hausdorff measure (U. Boscain and al. proved that the Hausdorff measure is not smooth in general in the contact case for $d \geq 5$).

- dW_{Δ} is supported by $S\Sigma$ (and even by $S\left(E^0_{r-1}\right)$ in the equi-regular case)

- The Weyl measure is often supported by the set where more brackets are needed (Grushin and Martinet cases) and hence can be singular w.r. to μ.

- If QE holds, the limit measure is dW_{Δ}.

  If support(dw_{Δ}) $= K \subset X$, then almost all eigenfunctions concentrate on K: $\int_X f|\phi_{n_j}|^2 d\mu \to 0$ if support(f)$\cap K = \emptyset$ for a density 1 subsequence.

Q(uantum)L(limits) and Q(uantum)E(rgodicity).–

A probability measure ν on $S^\star X$ is called a Q(uantum)L(imit) if there exists a sequence of eigenfunctions ϕ_{n_j} with $n_j \to \infty$ so that

$$\lim_{j\to\infty} \langle A\phi_{n_j} | \phi_{n_j} \rangle = \int_{S^\star X} a d\nu$$

for any ΨDO A of degree 0 with principal symbol a.

We say that the eigen-basis $(\phi_n)_{n\in\mathbb{N}}$ satisfies QE if there exists a density one sub-sequence (λ_{n_j}) of (λ_n) so that

$$\lim_{j\to\infty}\langle A\phi_{n_j}|\phi_{n_j}\rangle = \int_{S^\star X} a dW_\Delta$$

for any ΨDO A of degree 0 with principal symbol a.

Density 1 means that

$$\lim_{\lambda\to\infty}\frac{\#\{\lambda_{n_j}\leq\lambda\}}{N(\lambda)} = 1 \ .$$

The historical example of QE is due to A. Shnirelman (74') (proved later by Zelditch, YCdV).–

If (X, g) is a closed Riemannian manifold whose geodesic flow is ergodic, QE holds for any eigen-basis of the Laplace-Beltrami operator with ν the normalized Riemannian volume. This applies in particular if the curvature of (X, g) is < 0.

ЭРГОДИЧЕСКИЕ СВОЙСТВА СОБСТВЕННЫХ ФУНКЦИЙ

А. И. Шнирельман

Пусть M — гладкое компактное риманово многообразие (возможно, с краем), dx — риманов элемент объема на M, Δ — оператор Лапласа на M; $u_1, u_2, \ldots$ — собственные функции Δ с собственными значениями $\lambda_1^2, \lambda_2^2, \ldots$, $u_k\Big|_{\partial M}=0$, $\int\limits_M |u_k(x)|^2 dx=1$. Пусть $a(x)$ — произвольная непрерывная функция на M; что можно сказать о поведении при $k\to\infty$ величины $\int\limits_M a(x)|u_k(x)|^2dx$, т. е. о распределении по M «массы» функции $u_k(x)$?

Чтобы сформулировать наш результат, напомним некоторые понятия и обозначения. Пусть T^*M — расслоение ковекторов на M, т. е. многообразие пар (x, ξ), $x\in M$, $\xi\in T_x^*M$; S^*M — многообразие ковекторов $z=(x, \omega)\in T^*M$ таких, что $|\omega|=1$. В T^*M определен канонический элемент объема $dx\,d\xi$ (мера Лиувилля): тем самым на S^*M индуцируется элемент объема $dz=\frac{dx\,d\xi}{d\,|\xi|}$. Геодезический поток G_t: $T^*M\to T^*M$ задается уравнениями Гамильтона $\dot{x}=\frac{\partial H}{\partial\xi}$, $\dot{\xi}=-\frac{\partial H}{\partial x}$, $H(x, \xi)=|\xi|$. На ∂M происходит отражение по правилу $\xi=(\xi_t, \xi_n)\to(\xi_t, -\xi_n)$, где ξ_t — касательная, ξ_n — нормальная компоненты ξ на ∂M, т. е. «угол падения равен углу отражения». Поток G_t сохраняет меру $dx\,d\xi$ и гамильтониан $|\xi|$, поэтому S^*M переводится этим потоком в себя с сохранением меры dz. Наконец, пусть $P_s(k)=e^{-\lambda_k^2 s}\Big/\sum\limits_j e^{-\lambda_j^2 s}$ — распределение вероятности на $\mathbf{Z}_+$, зависящее от параметра $s>0$. Мы скажем, что подпоследовательность $\{u_{k_j}\}$ имеет плотность 1, если $\sum\limits_j P_s(k_j)\to 1$ при $s\to 0$; это определение несколько шире обычного.

Т е о р е м а 1. *Пусть поток G_t: $S^*M\to S^*M$ эргодичен относительно меры dz. Тогда найдется подпоследовательность $\{u_{k_j}\}$ плотности 1 такая, что для всякой непрерывной функции $a(x)$*

$$\int\limits_M a(x)\,|u_{k_j}(x)|^2\,dx\to\int\limits_M a(x)\,dx\Big/\int\limits_M dx \quad \text{при } j\to\infty.$$

Теорема 1 следует из более общей теоремы 2. Пусть $\hat{A}$ — псевдодифференциальный оператор (п. д. о.) порядка 0 на M с символом $A(x, \xi)$, $A(x, t\xi)=A(x, \xi)\,\forall t>0$ [1]. Пусть $A(z)$ — сужение $A(x, \xi)$ на S^*M. Из неравенства Гординга для п. д. о. [2] следует

Л е м м а 1. *На S^*M существует последовательность мер неотрицательных борелевских $\mu_k(dz)$ $(k=1, 2, \ldots)$, $\mu_k(S^*M)=1$, таких, что для всякого п. д. о. $\hat{A}$ порядка 0 с символом $A(x, \xi)$*

$$\left|(\hat{A}u_k, u_k)-\int\limits_{S^*M} A(z)\,\mu_k(dz)\right|\to 0 \quad \textit{при}\quad k\to\infty$$

(скалярное произведение берется в $L_2(M)$).

Т е о р е м а 2. *Если поток G_t эргодичен, то существует подпоследовательность $\{u_{k_j}\}$ плотности 1 такая, что для всякой непрерывной функции $A(z)$*

$$\int\limits_{S^*M} A(z)\,\mu_{k_j}(dz)\to\int\limits_{S^*M} A(z)\,dz\Big/\int\limits_{S^*M} dz \quad \textit{при}\quad j\to\infty.$$

Теорема 1 получается из теоремы 2, если положить $A(z)=A(x, \omega)=a(x)$; в этом случае

$$\int\limits_M a(x)\,|u_{k_j}(x)|^2\,dx=(\hat{a}u_{k_j}, u_{k_j})\sim\int\limits_{S^*M} A(z)\,\mu_{k_j}(dz)\sim$$

$$\sim\int\limits_{S^*M} A(z)\,dz\Big/\int\limits_{S^*M} dz=\int\limits_M a(x)\,dx\Big/\int\limits_M dx,$$

где $\hat{a}$ — оператор умножения на $a(x)$.

Доказательство теоремы 2 основано на следующих двух леммах.

Л е м м а 2. *Для всякой непрерывной функции $A(z)$*

$$\sum_k P_s(k)\int\limits_{S^*M} A(z)\,\mu_k(dz)\to\int\limits_{S^*M} A(z)\,dz\Big/\int\limits_{S^*M} dz \quad \textit{при}\quad s\to 0.$$

Таким образом, меры $\mu_k(dz)$ «в среднем» лебеговы.

Л е м м а 3. *Пусть ∂M состоит из конечного числа гладких кусков, пересекающихся трансверсально. Существует подпоследовательность $\{u_{k_j}\}$ плотности 1 такая, что для всякой непрерывной функции $A(z)$ и всякого $t_0>0$*

$$\left|\int\limits_{S^*M} A(G_t'(z))\,\mu_{k_j}(dz)-\int\limits_{S^*M} A(z)\,\mu_{k_j}(dz)\right|\to 0$$

при $j\to\infty$ равномерно по t при $|t|\leqslant t_0$. Если $\partial M=\varnothing$, то это справедливо для всей последовательности $\{u_k\}$.

Таким образом, меры μ_{k_j} асимптотически инвариантны относительно геодезического потока.

Примером многообразия M, удовлетворяющего условиям теоремы 2, служит, конечно, многообразие отрицательной кривизны [3]. Другой пример — области с рассеивающей (т. е. вогнутой) границей, а также с чересчур сильно фокусирующей границей, изученные Я. Г. Синаем и Л. А. Бунимовичем [4]—[6].

Автор признателен В. И. Арнольду, Я. Г. Синаю и В. Ф. Лазуткину за многочисленные стимулирующие беседы.

ЛИТЕРАТУРА

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Поступило в Правление общества 12 декабря 1973 г.

Dear Professor Kolin de Verdier!

I am very much thankful to You for Your report about my work in the seminar Bony. I think, it took a lot of work to reconstruct in detail the proofs. Your version is more beautiful and modern than my original one. It'll soon appear as an Appendix to the book of V. Lazutkin "KAM theory and approximation to eigenfunc-
" Springer, 1990. This

This result has been extended to many cases: manifolds with boundaries, discontinuous metrics, semi-classical Schrödinger operators, large regular graphs. To our knowledge, nothing was known before our work in the sR case.

Path to QE Theorems.–

One needs to determine the microlocal Weyl measure and to get a *vanishing Theorem for the variance*: if $\int_{S^\star X} a dW_\Delta = 0$, then

$$\mathrm{Var}_\Delta(A) := \lim_{\lambda\to\infty} \frac{1}{N(\lambda)} \sum_{\lambda_n \le \lambda} |\langle A\phi_n|\phi_n\rangle|^2 = 0 \ .$$

Usually the vanishing of the variance comes from ergodicity assumptions for a suitable dynamics preserving dW_Δ.

QL and QE in the 3D contact case.–

Let us start with a 3D closed manifold X with a smooth measure μ and an oriented *contact* distribution, i.e. $E = \ker \alpha$ with $\alpha \wedge d\alpha$ non vanishing. Let us give a smooth sR metric g on E. There exists a unique contact form β so that $d\beta(e_1, e_2) = 1$ for any positive orthonormal frame (e_1, e_2) of E for g. Let us denote by Z the *Reeb vector field* of β (i.e. $\beta(Z) = 1$, $d\beta(Z, .) = 0$). Then the Popp measure $dP = |\beta \wedge d\beta|$ is $Z-$invariant. The Weyl formula reads

$$N(\lambda) \sim \frac{\int_X dP}{32} \lambda^2 \ .$$

The Weyl measure is $dw_\Delta = \frac{1}{\int_X dP} dP$. Let us denote by Σ the symplectic sub-cone of $T^\star X$ generated by α. The sphere bundle $S\Sigma$ is a two-fold covering of X and and dW_Δ is one half of the pull back on $S\Sigma$ of dw_Δ.

An Hamiltonian interpretation of Z.–

$r : \Sigma \to \mathbb{R}$ is the positively homogeneous function with value 1 on $\pm\beta$, the Hamiltonian vector field $\mathcal{X}_r$ on Σ projects onto the Reeb vector field $\pm Z$.

Our results are the following [ArXiv 2015]:

Theorem 1 If the dynamics of the Reeb vector field Z is ergodic for the Popp volume, QE holds for any real eigenbasis of the sR Laplacians $\Delta_{g,\mu}$

Theorem 2 No assumption on the dynamics. Any QL ν splits $\nu = \nu_0 + \nu_1$ where

- **ν_0 is supported by $S\Sigma$ and invariant under the Reeb vector field**

- **$\nu_1(S\Sigma) = 0$ and ν_1 is invariant under the sR geodesic flow**

Example: X is the unit cotangent bundle of a 2D closed Riemannian manifold (M, h), $E = \ker \alpha$ with α the Liouville form. Then we can choose g so that the Reeb vector field is the geodesic flow of (M, h).

Question: in this case, what is the link of our result and Shnirelman's Theorem? Work in progress if M is an hyperbolic surface with Joachim Hilgert and Tobias Weich.

Main intuition: The Reeb dynamics and the geodesics

All sR geodesics in $T^\star X$ with Cauchy data $(x_0, \xi_0 + \tau\alpha(x_0)) \in T^\star X$ with $\tau \in \mathbb{R}$ have the same Cauchy data in TX. As $\tau \to \pm\infty$, they spiral around the trajectories of $\pm Z$. Reeb trajectories are C^0 limits of geodesics.

More precisely, we use a Birkhoff normal form along Σ already discussed by Melrose [84]. It has the following simple form

$$\Delta \equiv R \otimes \Omega + O_{\Sigma}(\infty)$$

along Σ where

- R is a Toeplitz elliptic operator of degree 1 associated to the symplectic cone Σ whose principal symbol is the Reeb Hamiltonian r,

- Ω is an harmonic oscillator.

Both operators commute and it allows to use the Reeb dynamics in the proof of the vanishing of the variance.

The Weyl asymptotics and the normal form.–

$$N_{\Delta}(\lambda) \sim \sum_{l=0}^{\infty} N_R\left(\frac{\lambda}{2l+1}\right) \text{ and } N_R(\mu) \sim \frac{V}{4\pi^2}\mu^2 ,$$

with V the Popp volume, give

$$N_{\Delta}(\lambda) \sim \frac{V\lambda^2}{4\pi^2} \sum_{l=0}^{\infty} \frac{1}{(2l+1)^2} ,$$

This gives

$$N_{\Delta}(\lambda) \sim \frac{V\lambda^2}{32} .$$

Remark: the Popp volume and the asymptotic linking invariant introduced by V. Arnold.

If $X = S^3$ and Z is a divergence free vector field, Arnold introduced an invariant measuring the average asymptotic linking number of two long trajectories of Z. In our case, this is exactly $1/\int_X dP$. Hence, Weyl formula shows that the Arnold invariant is a spectral invariant.

It is known [Melrose 1984] that the lengths of closed geodesics are determined by the spectrum. It is likely that they are family of closed geodesics spiraling around the closed Reeb orbits in a precise way, with lengths $T_n \sim \sqrt{2\pi k T}$.

On the other hand, if the normal form were exact, we would have:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} = \sum_{l=1}^{\infty} \frac{1}{(2l+1)^s} . \sum_{n=1}^{\infty} \frac{1}{\mu_n^s} \ ,$$

where the μ_n's are the eigenvalues of R. Apply then the Toeplitz version of the wave trace (Boutet-Guillemin).

This leads us to the

Conjecture 1: The Reeb periods are spectral invariants of Δ_{SR} in the 3D contact case.

The Grushin case.–

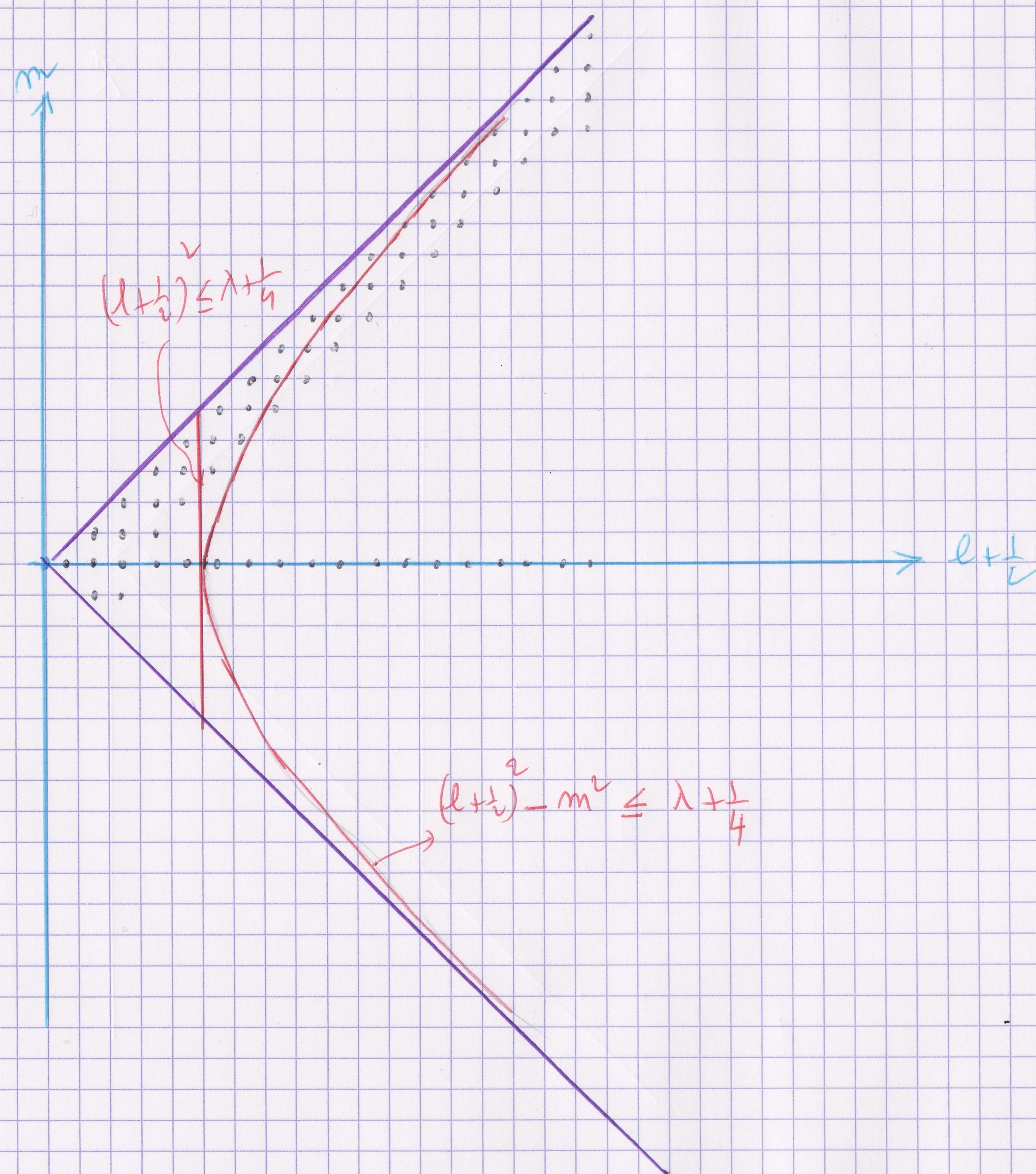
X is a 2D closed manifold and $\Delta = X_1^\star X_1 + X_2^\star X_2$ where X_1 and X_2 are independent outside of a 1D closed manifold Y: more precisely $X_1 \wedge X_2$ vanish on Z with a non zero differential and $([X_1, X_2], X_1, X_2)$ generate TX. Note that g is Riemannian outside of Y.

An explicit example.–

$X = S^2 \subset \mathbb{R}^3_{x_1,x_2,x_3}$. Let us denote by $X_1 = x_2\partial_{x_3} - x_3\partial_{x_2}, \ \cdots$. The Laplace Beltrami operator is $\Delta = -\left(X_1^2 + X_2^2 + X_3^2\right)$. The operator $\Delta_{sR} = -\left(X_1^2 + X_2^2\right)$ is a Grushin Laplacian commuting with X_3 and hence with Δ. Y is the equator $x_3 = 0$. If $(Y_{l,m})_{|m|\leq l}$ is the usual basis of spherical harmonics, we have

$$\Delta_{sR} Y_{l,m} = \left(l(l+1) - m^2\right) Y_{l,m}$$

The Weyl asymptotics is given by a lattice point problem for integer points between an hyperbola and his asymptotes. We get $N(\lambda) \sim \frac{1}{2}\lambda \ln \lambda$.



m
$\ell+\frac{1}{2}$
$(\ell+\frac{1}{2})^2 \leq \lambda+\frac{1}{4}$
$(\ell+\frac{1}{2})^2 - m^2 \leq \lambda+\frac{1}{4}$

Weyl asymptotics.–

Near Y, the Riemannian measure satisfies $dx_g \sim \left|\frac{dS}{S}\right| \otimes d\nu$ where S is any local equation of Y and ν a measure on Y.

We have the following 2-terms heat expansion

$$\int_M f(x)e(t,x,x)d\mu = \frac{1}{4\pi t}\left(|\ln t|\int_Y f d\nu + \mathbf{p.v.}_g \int_X f dx_g + A\int_Y f d\nu + o(1)\right)$$

with $A = \gamma + 4\ln 2$ and γ the Euler constant.

Note that this result CANNOT be obtained from the pointwise asymptotics of the heat kernels.

It follows that

$$N(\lambda) \sim \frac{\int_Y d\nu}{4\pi} \lambda \log \lambda$$

$$dw_\triangle = \frac{1}{\nu(Y)} d\nu$$

and $dW_\triangle$ is the lift to $S\Sigma$ (a twofold cover of Y) of $dw_\triangle$.

QE and QL.–

Theorem 3 *If Y is connected, QE holds.*

Every QL dm splits $dm = dm_0 + dm_1$ where dm_0 is supported by $S\Sigma$ and is a locally constant multiple of dW_Δ, $dm_1(S\Sigma) = 0$ and dm_1 is invariant under the (Riemannian) geodesic flow.

This follows from the desingularization which is contact 3D and our result on QL's in this case.

Martinet case.–

We consider the 3D case where $E = \ker \alpha$ with $\alpha \wedge d\alpha$ vanishing with a non zero differential on a sub-manifold $Y \subset X$ (locally $\alpha = dx - z^2 dy$). We assume, for simplicity, that E and TY are transversal at all points of Y. On $X \setminus Y$, we have a contact sR metric and a Popp volume dP. Locally, if Y is defined by $S = 0$, we have $dP = \left|\frac{dS}{S}\right| \otimes d\nu + 0(1)$ where $d\nu$ is a measure on Y. Similarly to the Grushin case, we have a 2-term heat expansion from which follows the Weyl asymptotics

$$\sum_{\lambda_n \leq \lambda} \int f \phi_n^2 d\mu \sim \frac{\int_Y f d\nu}{32} \lambda^2 \log \lambda \ .$$

Hence $dw_\Delta = (1/\nu(Y)) d\nu$ and dW_Δ is the lift to $S(\Sigma_Y)$.

This implies that a density 1 sub-sequence concentrates on Y.

This is another form of the results of R. Montgomery in the nice CMP (95) paper “Hearing the zero locus of a magnetic field”.

Conjecture 2: QE in the Martinet case

Let us denote by $F = E \cap TY$ the 1D foliation of Y induced by the distribution E. The manifold Y, if connected and orientable, is a 2 torus. Assume that F is isomorphic to a constant irrational foliation on $\mathbb{R}^2/\mathbb{Z}^2$. We conjecture is that QE holds in this case.

Thank you,

further progress is expected
from the ANR contract SRGI which starts this year.
Any other help will be welcome!