Embeddable 3d CR-manifolds Conference in memory of Louis Boutet de Monvel

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Acknowledgment

I want to thank the organizers for inviting to speak at this meeting in memory of one of my great heroes, Louis Boutet de Monvel. His work in several complex variables was both an inspiration to me and of great practical use.

The work I will describe is rather old, but I have not had many opportunities to discuss it lately. I appreciate your indulgence.

Outline

What is a CR-manifold?

For this audience, I scarcely need to review this, but it's always a good idea to start at the beginning: A complex structure on a $2n$ manifold, X, is a formally integrable, half-dimensional sub-bundle, $T^{0,1}X \subset TX \otimes \mathbb{C}$; if $\overline{Z}$ and $\overline{W}$ are local sections, then so is $[\overline{Z}, \overline{W}]$. We also require $TX \otimes \mathbb{C} = T^{0,1}X \oplus T^{1,0}X$. If n is at least 2, then this induces a CR-structure on a real hypersurface M in X, by setting

$$T^{0,1}M = T^{0,1}X \restriction_M \cap TM \otimes \mathbb{C}. \tag{1}$$

The fiber dimension of $T^{0,1}M$ is $n-1$, and it too is formally integrable. We have the splitting

$$TM \otimes \mathbb{C} = T^{0,1}M \oplus T^{1,0}M \oplus R, \tag{2}$$

where R is a 1-dimensional sub-bundle spanned by a real vector field T_R.

Strict Pseudoconvexity

The bundle $T^{0,1}M \oplus T^{1,0}M = H \otimes \mathbb{C}$, where H is a hyperplane bundle. If θ is a 1-form that locally defines H, then we say that $(M, T^{0,1}M)$ is strictly pseudoconvex is the Levi form

$$\mathcal{L}(\overline{Z}, \overline{W}) = d\theta(\overline{Z}, W) \tag{3}$$

is positive definite. This implies that H defines a contact structure on M. A case of special interest is when $M = \partial X$, in which case X is a modification of a Stein space.

Abstract CR-manifolds

As is well known, an abstract CR-manifold is defined as a $(2n-1)$-dimensional manifold M with a formally integrable, non-degenerate, sub-bundle $T^{0,1}M \subset TM \otimes \mathbb{C}$, of fibre dimension $n-1$. If the Levi form is positive definite, then we say that the CR-manifold is strictly pseudoconvex. The sub-bundle defines a differential operator, $\bar{\partial}_b$, by

$$\bar{\partial}_b f = df \restriction_{T^{0,1}M} . \tag{4}$$

If $M \subset X$ is a hypersurface in a complex manifold, and f is a holomorphic function defined in a neighborhood of M, $\bar{\partial}_b(f \restriction_M) = 0$, and more-or-less vice versa.

Given an abstract, compact strictly pseudoconvex CR-manifold M it is a natural question whether or not M can be realized as a hypersurface in a complex manifold, or more generally if it can embedded in $\mathbb{C}^N$ (for some N), so that the induced CR-structure is the given one.

If $\Phi = (\phi_1, \dots, \phi_N) : M \to \mathbb{C}^N$ is an immersion, then the induced CR-structure agrees with the given one if and only if

$$\bar{\partial}_b \phi_i = 0 \text{ for } i = 1, \dots, N. \tag{5}$$

We call such an embedding a CR-embedding; a CR-manifold that admits such an embedding is call *embeddable*.

Embeddability is a question as to whether the algebra, $\ker \bar{\partial}_b$, is large enough to embed M.

Boutet's Theorem on Embedding CR-manifolds

Boutet de Monvel essentially answered this question. I give a slightly different statement from that appearing in his paper, which is really just a Seminar Goulaouic–Schwartz lecture.

Theorem (Boutet de Monvel)

If $(M, T^{0,1}M)$ is compact strictly pseudoconvex CR-manifold, then there exists a CR-embedding M into $\mathbb{C}^N$, for some N, provided that the range of $\bar{\partial}_b$ is closed (in $L^2(M)$).

From earlier work of Kohn and Rossi it followed that range of $\bar{\partial}_b$ is closed whenever $\dim M \geq 5$, so this hypothesis is only meaningful in dimension 3.

The Harvey-Lawson Theorem

The image of the embedding $\Phi(M)$ is a "maximally complex" submanifold of $\mathbb{C}^N$, that is, the fiber dimension of $T^{0,1}\mathbb{C}^N\restriction_M \cap TM\otimes\mathbb{C}$ is constant and equal to $n-1$. Harvey and Lawson proved that there is always an analytic current V so that $\Phi(M)$ is the boundary of V.

With a little more work, one can show that there is a Stein space with isolated singularities, X, so that $\Phi(M)\simeq\partial X$, and the given CR-structure is induced by this inclusion. Under this hypothesis Kohn showed that the range of $\bar{\partial}_b$ is always closed.

Does anything actually go wrong in 3-dimensions?

Outline

The Closed Range Condition in 3d

The first indication that things go awry in 3d is an old example of Grauert, Rossi, Andreotti, etc. One starts with the unit 3-sphere in $\mathbb{C}^2$. The CR-structure is globally generated by the vector field $\overline{Z} = z_1\partial_{\bar{z}_2} - z_2\partial_{\bar{z}_1}$. On a 3-manifold the integrability condition is vacuous, so almost any complex vector field defines a CR-structure. In this case we consider the structures defined by

$$\overline{Z}_\epsilon = \overline{Z} + \epsilon Z. \tag{6}$$

It was shown that if $0 < |\epsilon| < 1$, then this CR-manifold cannot be realized as the boundary of any Stein surface. Indeed, Dan Burns showed that $\ker \overline{Z}_\epsilon$ contains only even functions, and hence cannot separate points on the sphere.

Nirenberg's Theorem

Indeed Nirenberg showed that if $(M, T^{0,1}M)$ is an embeddable 3d-CR manifold, with CR-structure generated locally by $\overline{Z}$, then for a generic function φ, with arbitrarily small norm, the local solutions to $(\overline{Z} + \varphi Z)u = 0$ are constants. Hence, embeddability is highly unstable!

In this lecture we consider the properties of the set of deformations of an embeddable CR-structure, which are themselves embeddable.

What goes wrong

It is easiest to describe what goes wrong in terms of the spectrum of the $\Box_b$-operator, where $\Box_b = \bar{\partial}_b^* \bar{\partial}_b$.

1. If the structure is embeddable, then the spectrum of $\Box_b$ is discrete, with 0 an eigenvalue of infinite multiplicity.
2. If the structure is not embeddable, then the eigenvalue at zero splits up into and infinite sequence $\{\mu_j\}$ that satisfies $\mu_j = O(j^{-N})$, for any N.
3. Under an embeddable deformation the eigenvalue at zero can also split off a finite number of "small" eigenvalues.

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Some Notation

The deformations of the CR-structure are parametrized by

$$\mathfrak{D}(M, T^{0,1}M) = \{\psi \in \mathscr{C}^\infty(M; \operatorname{Hom}(T^{0,1}M, T^{1,0}M)) : \|\psi\|_\infty < 1\}. \tag{7}$$

The deformed structure is the graph

$${}^{\psi}T_p^{0,1}M = \{\overline{Z} + \psi_p(\overline{Z}) : \overline{Z} \in T_p^{0,1}M.\}. \tag{8}$$

The group of contact diffeomorphisms acts on this representation, with orbits representing geometrically equivalent structures. We let $\mathscr{E}(M, T^{0,1}M) \subset \mathfrak{D}(M, T^{0,1}M)$ denote the embeddable structures.

Outline

The Embeddable Structures

Even modulo the contact action, the subset $\mathscr{E}(M, T^{0,1}M)$ is both of infinite dimension and infinite co-dimension. Questions that we therefore want to answer are:

1. Is $\mathscr{E}(M, T^{0,1}M)$ a closed subset of $\mathscr{D}(M, T^{0,1}M)$?
2. Is $\mathscr{E}(M, T^{0,1}M)$ a connected set?
3. Does $\mathscr{E}(M, T^{0,1}M)$ have the structure of a "manifold" or a fibered space over a finite dimensional base?
4. How do you tell if a given deformation is embeddable?

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Some Answers

Today we can give a satisfactory answer to the first question. The others can only be answered in special cases. The key insight is that in the case of the structures $\overline{Z}_\epsilon$, in some sense

$$\dim\left(\frac{\ker \overline{Z}}{\ker \overline{Z}_\epsilon}\right) = \infty. \tag{9}$$

To make this more meaningful, we let $S : L^2(S^3) \to \ker \overline{Z}$ denote the Szegő projector, which is an orthogonal projector onto the null-space of $\overline{Z}$. For small ϵ one can show that $S \restriction_{\ker \overline{Z}_\epsilon}$ is injective and indeed we have

$$\dim\left(\frac{\ker \overline{Z}}{S \ker \overline{Z}_\epsilon}\right) = \infty. \tag{10}$$

The question we want to answer is ultimately a question about the stability of the $\ker \bar{\partial}_b$ under deformations.

The Relative Index

To quantitatively assess this stability we need a way to compare the null-spaces of $\bar{\partial}_b$ for different CR-structures. The Szegő projector provides a way to do this. Let $(M, T^{0,1}M)$ be an embeddable 3d-CR manifold, with $\bar{\partial}_b$ the associated operator, and S an orthogonal projection onto $\ker \bar{\partial}_b$. If $\bar{\partial}_b^{\psi}$ is the analogous operator defined by a deformation, ψ, of this CR-structure, then we consider the restriction:

$$S : \ker \bar{\partial}_b^{\psi} \longrightarrow \ker \bar{\partial}_b. \tag{11}$$

One can show that if $\psi \in \mathcal{E}$, if and only if this restriction is Fredholm operator, that is, it has a finite dimensional kernel and co-kernel. We define the *relative index*,

$$\text{R-Ind}(\bar{\partial}_b, \bar{\partial}_b^{\psi}) = -\left[\dim\ker(S \restriction_{\ker \bar{\partial}_b^{\psi}}) - \dim\left(\frac{\ker \bar{\partial}_b}{S \ker \bar{\partial}_b^{\psi}}\right)\right].$$

Extension of the Szegő Projector

Many remarkable properties follow from the following observation: Let Q be the partial inverse to $\bar{\partial}_b^* \bar{\partial}_b$, that is

$$Q\bar{\partial}_b^* \bar{\partial}_b = \mathrm{Id} - S. \tag{12}$$

If $\psi \in \mathfrak{D}(M, T^{0,1}M)$, then

$$S \restriction_{\ker \bar{\partial}_b^{\psi}} = (\mathrm{Id} + Q\bar{\partial}_b^* \psi \circ \bar{\partial}_b) \restriction_{\ker \bar{\partial}_b^{\psi}} . \tag{13}$$

This formula is very useful because the operator $(\mathrm{Id} + Q\bar{\partial}_b^* \psi \circ \bar{\partial}_b) : L^2(M) \to L^2(M)$, is Fredholm whenever $\|\psi\|_\infty < 1$. Moreover for small enough deformations it is automatically invertible, hence, for small enough deformations, ψ,

$$\text{R-Ind}(\bar{\partial}_b, \bar{\partial}_b^{\psi}) \geq 0. \tag{14}$$

Functorial Properties of the Relative Index

If both structures are embeddable, then we can reverse their roles. It is easy to see that

$$\text{R-Ind}(\bar{\partial}_b, \bar{\partial}_b^{\psi}) = -\text{R-Ind}(\bar{\partial}_b^{\psi}, \bar{\partial}_b). \tag{15}$$

In fact, if we have three embeddable deformations, $\bar{\partial}_b^1, \bar{\partial}_b^2, \bar{\partial}_b^3$, then we have a co-cycle formula:

$$\text{R-Ind}(\bar{\partial}_b^1, \bar{\partial}_b^3) = \text{R-Ind}(\bar{\partial}_b^1, \bar{\partial}_b^2) + \text{R-Ind}(\bar{\partial}_b^2, \bar{\partial}_b^3). \tag{16}$$

The proof of this general result, which appears in a paper I wrote with Melrose, was provided by Louis. To prove it we extend the notion of Szegő projector to generalized Szegő projectors, which are microlocally like Szegő projectors. This concept had been introduced earlier by Guillemin and Boutet.

The Stratification

We define a stratification of $\mathcal{E}(M, T^{0,1}M)$ by setting

$$\mathfrak{S}_n = \{\psi \in \mathcal{E}(M, T^{0,1}M) : \text{R-Ind}(\bar{\partial}_b, \bar{\partial}_b^{\psi}) \leq n\}. \tag{17}$$

Using the formula

$$S \restriction_{\ker \bar{\partial}_b^{\psi}} = (\mathrm{Id} + Q\bar{\partial}_b^{*}\psi \circ \bar{\partial}_b) \restriction_{\ker \bar{\partial}_b^{\psi}}, \tag{18}$$

one can show that these strata are closed sets.

A very optimistic person might conjecture that there exists an N so that

$$\mathcal{E}(M, T^{0,1}M) \subset \mathfrak{S}_N, \tag{19}$$

at least for sufficiently small deformations.

Evidence that this Might be True

There was evidence that this might, in fact, be true.

1. Using further properties of the relative index, and a remarkable result of Eliashberg, one can show that

$$\mathcal{E}(S^3, T^{0,1}S^3) = \mathfrak{S}_0. \tag{20}$$

2. If $M \to \Sigma^g$ is a circle bundle over a Riemann surface of degree $d \geq 3g - 3$, then I showed that sufficiently small perturbations have bounded relative index.
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Outline

The Relative Index Formula, I

The suggestion that there might in fact be a reasonable formula for the relative index was suggested to me by Laszlo Lempert. To obtain it however requires a pretty considerable excursion away from several complex variables.

Lempert was aware of the very substantial literature on relative indices in the context of elliptic boundary value problems for Dirac operators on manifolds with boundary. These ideas go back to Atiyah, Bott, Patodi, Singer, etc. There was also a conjecture related to this of Atiyah and Weinstein.

From my perspective Spin-C structures and Dirac operators is the correct framework in which to consider this question.

The Complex Case

If X is a complex manifold with strictly pseudoconvex boundary, then the bundle

$$\not{S} = \oplus_{q=1}^{n} \Lambda^{0,q} X \tag{21}$$

is a Spin-C bundle, and $\eth = \bar{\partial} + \bar{\partial}^*$ is a Spin-C Dirac operator.

The usual analysis of these operators employs boundary conditions defined by spectral projectors defined by a elliptic boundary operator. These define the standard elliptic boundary conditions that appear in the work of APS, but have nothing to do with complex analysis.

To make the connection to complex analysis one needs to use the $\bar{\partial}$-Neumann condition. The difficulty is that it does not define a Fredholm operator because the null-space contains holomorphic functions and is therefore infinite dimensional.

The $\bar{\partial}$-Neumann Condition

If ρ is a defining function for the ∂X, and $\alpha = \alpha^{0,0} + \cdots + \alpha^{0,n}$ is a spinor, then the $\bar{\partial}$-Neumann condition is that

$$\bar{\partial}\rho \rfloor \alpha^{0,q} = 0 \text{ for } 0 < q. \tag{22}$$

There is no condition if $q = 0$, which is what allows all holomorphic functions to belong to the null-space.

The Modified $\bar{\partial}$-Neumann Condition

To get a Fredholm problem we must therefore modify the boundary condition for $q = 0$.

Let S be the Szegő projector defined on ∂X. To get a formally self-adjoint, sub-elliptic boundary condition we replace the conditions for $q = 0, 1$ with

$$S(\alpha^{0,0} \restriction_{\partial X}) = 0 \text{ and } (\mathrm{Id} - S)[\bar{\partial}\rho \rfloor \alpha^{0,q} \restriction_{\partial X}] = 0. \tag{23}$$

This boundary condition is formally self adjoint, and sub-elliptic, and therefore $\eth$ with this boundary condition is a Fredholm operator. We denote the operator defining the boundary condition by $\mathcal{R}_S$. As usual we split the Dirac operator into its even and odd parts, $\eth^{\mathrm{eo}}$.

The Hard Part

The difficult fact needed to solve this problem is to show that

$$[\eth^{eo}]^* = \overline{\eth^{oe}}. \tag{24}$$

The main step is to construct a parametrix for the boundary value problem, which is reduced to analysis on the boundary.

To do this analysis I use the extended Heisenberg calculus, which I developed with Melrose. This calculus contains both the classical calculus of pseudodifferential operators and the Heisenberg calculus of Beals, Greiner, Taylor,.... as sub-algebras.

We were not the first to do this, but our formulation is especially clean for the purpose of symbolic computations. The Hermite calculus of Boutet and Victor is an ideal in the usual Heisenberg calculus.

The Index Formula in the Integrable Case

In the integrable case we have an explicit formula for the index:

$$\mathrm{Ind}(\eth^{\mathrm{e}}, \mathcal{R}_S) = \sum_{q=1}^{n} \dim H^{0,q}(X)(-1)^q \stackrel{d}{=} \chi'_{\mathcal{O}}(X). \tag{25}$$

This is not entirely trivial because the boundary condition is modified in degree 1 as well as degree 0.

More General Boundary Conditions

The projector S used to define the modified boundary condition does not need to be connected in any way to the complex structure. Indeed, we do not actually need an integrable complex structure or a genuine Szegő projector to define either the operator, $\eth$, or the boundary condition, $\mathcal{R}_S$.

The Spin-C Case

In fact, this all generalizes to the case that X is a Spin-C manifold with boundary, provided that there is a neighborhood of the boundary, U in which the Spin-C structure is defined by an almost complex structure J. We need to assume that $\bar{\partial}_J \rho \restriction_{\rho=0}$ defines a contact form with respect to which the boundary is strictly pseudoconvex. In this case

$$\not{S} \restriction_U \simeq \bigoplus_{q=0}^{n} \Lambda_J^{0,q} X. \tag{26}$$

Moreover, S can be any generalized Szegő projector. The modified $\bar{\partial}$-Neumann condition still makes sense, and $(\eth, \mathcal{R}_S)$ is again a (sub-elliptic) Fredholm operator. Of course in this generality we do not have a simple explicit formula for the index of $(\eth^{\mathrm{e}}, \mathcal{R}_S)$.

Agranovich-Dynin Formula

If S and S' two generalized Szegő projectors, which define boundary conditions $\mathcal{R}_S$ and $\mathcal{R}_{S'}$, then the indices of the associated operators satisfy:

$$\text{R-Ind}(S, S') = \text{Ind}(\eth^{\text{e}}, \mathcal{R}_S) - \text{Ind}(\eth^{\text{e}}, \mathcal{R}_{S'}). \tag{27}$$

This generalizes a classic formula, known as the Agranovich-Dynin formula, where the boundary conditions are defined by classical pseudodifferential projectors, and the boundary value problems are elliptic.

Proof of the Agranovich-Dynin Formula

To prove this formula we again restrict to the boundary. The key is to show that $= \operatorname{Ind}(\eth^{e}, \mathcal{R}_S)$ is the "relative index" of $\mathcal{R}_S$ and $\mathcal{P}_{\eth}$, where the later projection is the Calderon projection of $\eth^{e}$.

This is not entirely trivial because these projectors do not define a Fredholm pair in a standard sense. To prove this result we introduce a notion of "tame Fredholm pair," for which we can define an index, and also give a formula for the index in terms of traces of residual terms. (The operator $RP + (I - R)(I - P)$ is "tamely elliptic.")

Using this formula we can also prove a the usual logarithmic formula for compositions of tame Fredholm pair, which leads to a proof of the Agranovich-Dynin formula.

Gluing in the Spin-C Category

The reason I prefer the Spin-C category to the almost complex category is that it is easier to glue Spin-C manifolds along their boundaries, since there is no intrinsic notion of convexity. Boutet de Monvel, Leichtnam, and Zhang did something similar, but stayed in the almost complex category. This required them to work with non-Hausdorff spaces.

If $(X_1, \not{S}_1)$ and $(X_2, \not{S}_2)$ are Spin-C manifolds, as above with contact equivalent boundaries, denoted by Y, then on the glued space

$$X_{12} = X_1 \amalg_Y \overline{X_2}, \tag{28}$$

($\overline{X_2}$ is X_2 with orientation reversed), there is a glued Spin-C bundle

$$\not{S}_{12} = \not{S}_1 \amalg_Y \overline{\not{S}_2}. \tag{29}$$

Here $\overline{\not{S}_2}$ is $\not{S}_2$ with even and odd spinors interchanged.

The Relative Index Formula

If $(X_1, \mathcal{S}_1)$ and $(X_2, \mathcal{S}_2)$ are as above ($\partial X_1 \simeq \partial X_2$ as contact manifolds), and S_1, S_2 are generalized Szegő projectors as above, then we have the following formula for the relative index:

$$\text{R-Ind}(S_1, S_2) = \text{Ind}(\eth^{e}_{X_{12}}) - \text{Ind}(\eth^{e}_{X_1}, \mathcal{R}_{S_1}) + \text{Ind}(\eth^{e}_{X_2}, \mathcal{R}_{S_2}). \quad (30)$$

Of course, in this generality there is very little hope of saying anything definite about either side of this formula. But is special cases it can be made quite explicit. If X_1, and X_2 are complex manifolds, and the Szegő projectors are those defined by the complex structure, then

$$\text{R-Ind}(S_1, S_2) = \text{Ind}(\eth^{e}_{X_{12}}) - \chi'_{\mathcal{O}}(X_1) + \chi'_{\mathcal{O}}(X_2). \quad (31)$$

There is a cohomological formula for $\text{Ind}(\eth^{e}_{X_{12}})$.

Outline

The 3d-case

In the 3d-case this formula can be made quite explicit:

$$\text{R-Ind}(S_1, S_2) = \dim H^{0,1}(X_1) - \dim H^{0,1}(X_2) + \frac{\operatorname{sig}([X_1]) + \chi([X_1]) - \operatorname{sig}([X_2]) - \chi([X_2])}{4}. \quad (32)$$

Here $\operatorname{sig}(X_i)$ and $\chi([X_i])$ are the signature of the intersection pairing on $H_2(X_i)$ and the Euler characteristic, respectively. To prove the relative index conjecture we need to carefully analyze this formula.

What is needed?

Recall that we want to show that R-Ind(S_1, S_2) is bounded above. All the difficulty comes from the topological term.

If it were the case that only finitely many homotopy types can occur as the Stein filling of strictly pseudoconvex 3d-contact manifold, then it would follow that these numbers only assume finitely many values.

However, it is now known that it possible for infinitely many different topological types to occur as the Stein filling of certain 3d-contact manifolds.

The Topological Invariants

We let $b_j(Z) = \dim H_i(Z)$ denote the Betti numbers. We let Y denote the common boundary of X_1 and X_2. These manifolds are homotopic to 2d-cell complexes, so that

$$\chi(X_i) = b_2(X_i) - b_1(X_i) + 1. \tag{33}$$

It is also a simple matter to show that

$$b_1(X_i) \leq b_1(Y). \tag{34}$$

The second Betti number is split into 3 parts: $b_2(X_i) = b_2^+(X_i) + b_2^-(X_i) + b_2^0(X_i)$, so that

$$\operatorname{sig}(X_i) = b_2^+(X_i) - b_2^-(X_i). \tag{35}$$

The Refined Formula

We these preliminaries, we can rewrite $\text{R-Ind}(S_1, S_2) = C_1 - C_2$, where

$$C_i = \dim H^{0,1}(X_i) + \frac{2b_2^+(X_i) + b_2^0(X_i) - b_1(X_i)}{4}. \tag{36}$$

From this it follows easily that

$$\text{R-Ind}(S_1, S_2) \leq C_1 + \frac{b_1(Y)}{4}, \tag{37}$$

which is what we wanted to prove. We have the following corollary:

Corollary

If $(M, T^{0,1}M)$ is a compact, embeddable, 3d, strictly pseudoconvex CR-manifold, then the set $\mathcal{E}(M, T^{0,1}M)$ is closed in the $\mathcal{C}^\infty$-topology.

Thanks!

Thanks for your attention!

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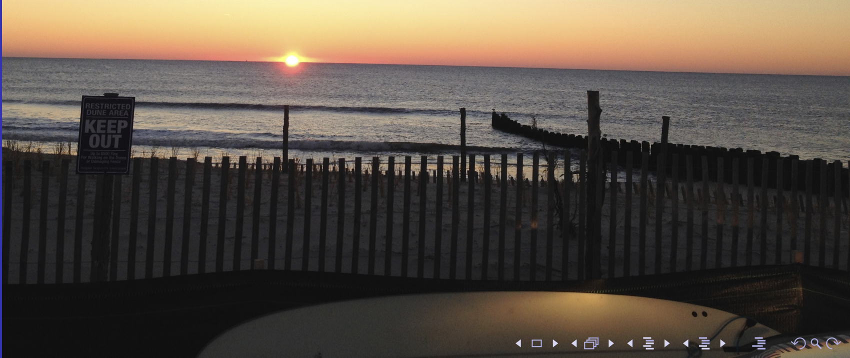

Outline

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