Dispersion and controllability for linear Schrödinger equations on compact Riemannian manifolds

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The Schrödinger equation

$$(S) \qquad i\frac{\partial u}{\partial t} = \left(\frac{\Delta}{2} + V\right) u$$

$$u_{\rceil t=0} = u^0$$

on a compact Riemannian manifold M.

Two questions

- dispersive properties
- observability : under which condition(s) on the open set $\Omega \subset M$ and on $T > 0$ can we prove an inequality of the type

$$(\mathrm{Obs}(\Omega, T)) \quad \|u^0\|^2_{L^2(M)} \leq C(T, \Omega) \int_0^T \|u(t)\|^2_{L^2(\Omega)} dt$$

for all u^0 ?

Microlocal point of view : take $(u_n^0)_{n\in\mathbb{N}}$ a sequence of initial conditions with

$$\|u_n^0\|_{L^2(M)} = 1,$$

and denote by $u_n(x,t)$ the corresponding solution.

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We study the "Wigner transforms",

$$W_{u_n}(a) = \langle u_n, a(x, D_x, t, D_t) u_n \rangle_{L^2(M\times\mathbb{R})}.$$

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Note that if $a = a(x,t)$, this is just

$$W_{u_n}(a) = \int_{M\times\mathbb{R}} a(x,t)|u_n(x,t)|^2 dxdt.$$

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$$W_n(a) = \langle u_n, a(x, D_x, t, D_t) u_n \rangle_{L^2(M\times\mathbb{R})}.$$

The symbol $a(x, \xi, t, H)$ is C_c^∞ in (x, t), C^∞ in (ξ, H), and satisfies the homogeneity condition

$$a(x, \xi, t, H) = a(x, \lambda\xi, t, \lambda^2 H)$$

for $\lambda > 1$ and $\|\xi\|^2 + |H| > R_0$.

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for $\lambda > 1$ and $\|\xi\|^2 + |H| > R_0$.
In other words, outside a compact set,

$$a(x, \xi, t, H) = a_{hom}(x, \xi, t, H)$$

where a_{hom} is homogeneous.

We denote by $\mathcal{S}^0$ this class of symbols.
There exists a subsequence such that, for all $a \in \mathcal{S}^0$,

$$W_{n_k}(a) \longrightarrow \mu(a)$$

for some $\mu \in (\mathcal{S}^0)'$ (microlocal defect measure).

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The "dispersion properties" we hope for are of the type "μ has some regularity / cannot be carried by a set that is too small".

$(\mathrm{Obs}(\Omega, T))$ can be deduced from the property "$\mu(\Omega \times (0, T)) > 0$" (for any sequence (u_n^0)).

Propagation of singularities

If $u_n^0 \longrightarrow 0$ weakly in $L^2(M)$, μ is a probability measure, supported on the sphere at infinity in the variables (ξ, H).
We can write

$$\mu(a) = \mu(a_{hom})$$

for all $a \in \mathcal{S}^0$.
Equation (S) implies that μ is carried on the set $\{\|\xi\|^2 = 2H\}$, and satisfies

$$\frac{\xi}{\sqrt{2H}} \cdot \partial_x \mu = 0.$$

(invariance under the geodesic flow)

More precisely,

$$\mu(dx, d\xi, dt, dH) = \mu_t(dx, d\xi)dt\, \delta_{H=\|\xi\|^2/2}$$

where, for almost every t, μ_t is a probability measure on the sphere bundle S^*M, invariant under the geodesic flow.

The geometric control condition GCC

Let $\Omega \subset M$. If every geodesic eventually enters Ω, then

$$\mu(\Omega \times (0, T)) > 0$$

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An argument of Bardos-Lebeau-Rauch (for V independent of t), then shows that $(\mathrm{Obs}(\Omega, T))$ holds for all T, if Ω satisfies (GCC).

GCC is a sufficient condition for observability, not a necessary one.

Negative curvature : the Quantum Unique Ergodicity conjecture

On a negatively curved manifold, this conjecture predicts that all sequences of initial conditions (u_n^0) lead to the same microlocal defect measure,

$$\mu = dx d\sigma(\xi) dt \, \delta_{H=\|\xi\|^2/2}$$

(for any sequence of initial conditions (u_n^0) converging weakly to 0).

But it's not even proven that such a measure μ charges every open set, or that it is absolutely continuous in x.

A-Nonnenmacher 2006, A-Rivière 2009

Denote $d = dim\, M$, and say $K \equiv -1$.
Let (u_n^0) be a sequence of initial conditions converging weakly to 0, let μ be a limit of the Wigner transforms W_{u_n}.

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Theorem

Then for a.e. t, the entropy of μ_t is $\geq \frac{d-1}{2}$. As a consequence, the Hausdorff dimension of the support of μ_t is $\geq d$.

In terms of observability, this implies the following. Let $\Omega \subset M$, and let $K \subset S^*M$ be the set of vectors (x, ξ) that generate geodesics that never enter Ω.

Theorem

Assume V does not depend on t.
If $\dim K < d$, then

$$\|u^0\|^2_{L^2(M)} \leq C(T, \Omega) \int_0^T \|u(t)\|^2_{L^2(\Omega)} dt$$

for all u^0 (for any T).

A-Macià 2010 : flat tori

Take for instance $\mathbb{T}^d = \mathbb{R}^d / 2\pi\mathbb{Z}^d$.
The geodesic flow is

$$\phi^s : (x, \xi) \mapsto (x + s\xi, \xi)$$

for $(x, \xi) \in \mathbb{T}^d \times \mathbb{R}^d$, $s \in \mathbb{R}$.

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The tori $\mathbb{T}_{\xi_0} = \{(x, \xi), \xi = \xi_0\}$ are preserved.

The properties of the trajectories on $\mathbb{T}_{\xi_0}$ depend on the rank of

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If $Rk\,\Lambda_{\xi_0} = 0$, trajectories are dense (and equidistributed) in $\mathbb{T}_{\xi_0}$.
If $Rk\,\Lambda_{\xi_0} = d-1$, trajectories are periodic in $\mathbb{T}_{\xi_0}$.

More generally, if $Rk\,\Lambda_{\xi_0} = d - j$, each trajectory $(x_0 + t\xi_0)_{t\in\mathbb{R}}$ fills densely a subtorus of dimension j, $x_0 + \mathbb{T}_{\xi_0}$, where we define

$$\mathbb{T}_{\xi_0} = \Lambda_{\xi_0}^{\perp}/(\Lambda_{\xi_0}^{\perp} \cap \mathbb{Z}^d).$$

Let Λ be a subgroup of $\mathbb{Z}^d$ (primitive, in the sense that $Vect(\Lambda) \cap \mathbb{Z}^d = \Lambda$). We denote

$$R_\Lambda = \{\xi, \Lambda_\xi = \Lambda\} \subset \mathbb{R}^d$$

We partition the phase space :

$$\mathbb{T}^d \times (\mathbb{R}^d \setminus \{0\}) = \sqcup_{\Lambda \neq \mathbb{Z}^d} \mathbb{T}^d \times R_\Lambda.$$

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In dimension $d = 2$, $\Lambda = 0$ corresponds to irrational slopes, and the other Λs correspond to closed geodesics.

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In dimension $d = 2$, $\Lambda = 0$ corresponds to irrational slopes, and the other Λs correspond to closed geodesics. Thus, if μ is a positive Radon measure on $\mathbb{T}^d \times (\mathbb{R}^d \setminus \{0\})$, we can write

$$\mu = \sum_{\Lambda \neq \mathbb{Z}^d} \mu \rceil_{\mathbb{T}^d \times R_\Lambda}.$$

Starting again from a normalized sequence (u_n^0) in $L^2(\mathbb{T}^d)$, denote by $u_n(x,t)$ the corresponding solution of (S), and

$$\mu(dx, d\xi, dt, dH) = \mu_t(dx, d\xi) dt \delta_{H=\|\xi\|^2/2}$$

a limit point of the sequence of Wigner transforms W_{u_n}.
Assume first that (u_n^0) converges weakly to 0 in $L^2(\mathbb{T}^d)$.
We apply the previous remark to each μ_t,

$$\mu_t = \sum_{\Lambda \neq \mathbb{Z}^d} \mu_t \rceil_{\mathbb{T}^d \times R_\Lambda}.$$

Study of $\mu_t\rceil_{\mathbb{T}^d \times R_\Lambda}$.

If $\Lambda = 0$, then invariance under there geodesic flow implies that

$$d\mu_t\rceil_{\mathbb{T}^d \times R_0} = dx \otimes d\nu_0(\xi).$$

If $Rk\,\Lambda = 1$, then invariance under there geodesic flow implies that $\mu_t\rceil_{\mathbb{T}^d \times R_\Lambda}$ is the Lebesgue measure in the direction of the hyperplane $x \in \Lambda^\perp$. There is only 1 non trivial direction to understand.

Theorem

If $Rk\,\Lambda = 1$*, we show the following : for all* $a \in \mathcal{S}^0$*,*

$$\int a d\mu_t\rceil_{\mathbb{T}^d \times R_\Lambda} = \int a_{hom} dx \otimes d\nu_\Lambda(\xi) + \int_{\xi \in R_\Lambda} \mathrm{Tr}_{L^2(\mathbb{T}_\Lambda)} \left(M_{\langle a_{hom} \rangle_{\Lambda^\perp}}(t,\xi) \rho_\Lambda(t,\xi) \right) d\ell_\Lambda(\xi)$$

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In addition (if V *is* C^0*), the operator* $\rho_\Lambda(t,\xi) \in \mathcal{L}^1_+(L^2(\mathbb{T}_\Lambda))$ *satisfies the Heisenberg propagation law*

$$\rho(t) = U^*_\Lambda(t)\rho(0)U_\Lambda(t)$$

where $U_\Lambda(t)$ *is the unitary propagator for the S-equation on* $L^2(\mathbb{T}_\Lambda)$*,*

$$(\mathrm{S}_\Lambda) \qquad i\frac{\partial v}{\partial t} = \left(\frac{\Delta_\Lambda}{2} + \langle V\rangle_{\Lambda^\perp}\right).$$

Second microlocalisation (2-microlocal defect measures)

The decomposition

$$\int a d\mu_t\rceil_{\mathbb{T}^d \times R_\Lambda} = \int a_{hom} dx \otimes d\nu_\Lambda(\xi) + \int_{\xi \in R_\Lambda} \mathrm{Tr}_{L^2(\mathbb{T}_\Lambda)} \left(M_{\langle a_{hom} \rangle_{\Lambda^\perp}}(t,\xi) \rho_\Lambda(t,\xi) \right) d\ell_\Lambda(\xi)$$

comes from a second microlocalisation :

$$\begin{aligned} W_{u_n}(a) &= \langle u_n, a(x, D_x, t, D_t) u_n \rangle_{L^2(\mathbb{T}^d \times \mathbb{R})} \\ &= \left\langle u_n, a(x, D_x, t, D_t)(1-\chi)\left(\frac{D_x^\Lambda}{R}\right) u_n \right\rangle_{L^2(\mathbb{T}^d \times \mathbb{R})} \\ &+ \left\langle u_n, a(x, D_x, t, D_t)\chi\left(\frac{D_x^\Lambda}{R}\right) u_n \right\rangle_{L^2(\mathbb{T}^d \times \mathbb{R})} \end{aligned}$$

and by taking the limits $n \longrightarrow \infty$ followed by $R \longrightarrow \infty$.

Theorem

More generally, we show that μ_t has the following structure : for all $a \in \mathcal{S}^0$,

$$\int a d\mu_t = \int a_{hom} dx \otimes d\nu(\xi) + \sum_{\Lambda \neq \mathbb{Z}^d} \int_{\xi \in \Lambda^\perp} \mathrm{Tr}_{L^2(\mathbb{T}_\Lambda)} \left(M_{\langle a_{hom} \rangle_{\Lambda^\perp}}(t,\xi) \rho_\Lambda(t,\xi) \right) d\ell_\Lambda(\xi).$$

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If V is continuous, each $\rho_\Lambda(t, \xi)$ satisfies the Heisenberg propagation law, for the Schrödinger equation (S_Λ) on $L^2(\mathbb{T}_\Lambda)$.

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If V is continuous, each $\rho_\Lambda(t,\xi)$ satisfies the Heisenberg propagation law, for the Schrödinger equation (S_Λ) on $L^2(\mathbb{T}_\Lambda)$. If (u_n^0) does not converge weakly to 0, there is an additional term

$$\mathrm{Tr}_{L^2(\mathbb{T}^d)}(a(x, D_x, t)\rho(t))$$

where $\rho(t)$ is the limit, for the weak-$\star$ topology of $\mathcal{L}^1_+(L^2(\mathbb{T}^d))$, of the rank-one projectors $|u_n(t)\rangle\langle u_n(t)|$.

Corollary

(absolute continuity) $V \in L^\infty(\mathbb{T}^d \times \mathbb{R})$

Let $\pi : \mathbb{T}^d \times \mathbb{R}^d \longrightarrow \mathbb{T}^d$. *Then* $\pi_* \mu_t$ *is absolutely continuous.*

Corollary

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Comes from the fact that measures of the form

$$a \mapsto \mathrm{Tr}(a(x)\widetilde{\rho})$$

for some $\widetilde{\rho} \in \mathcal{L}^1_+(L^2(\mathbb{T}_\Lambda))$ are absolutely continuous.

Corollary

(observability) $V \in C^0(\mathbb{T}^d \times \mathbb{R})$
Let $\Omega \subset \mathbb{T}^d$ *and, for any* $\Lambda \subset \mathbb{Z}^d$, *let* Ω_Λ *be its orthogonal projection on* $\mathbb{T}_\Lambda$.
Assume one the the following statements holds :

- *Unique continuation for all the equations* (S_Λ) *on* $\Omega_\Lambda \times (0, T)$
- *OR : the potential* $V \in C^0(\overline{\mathbb{D}}; \mathbb{R})$ *does not depend on* t.

Then the observability inequality holds :

$$(\mathrm{Obs}(\Omega, T)) \quad \|u^0\|^2_{L^2(M)} \leq C(T, \Omega) \int_0^T \|u(t)\|^2_{L^2(\Omega)} dt$$

for all u^0.

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Unique continuation is known to hold for any open set if $V(x,t)$ is analytic (Holmgren). If $V \in C^0$ does not depend on t, we can use an argument of Lebeau relying on unique continuation of eigenfunctions to show $(\mathrm{Obs}(\Omega, T))$ directly.

Reference to other work :

Absolute continuity of μ : for $V = 0$ and (u_n^0) eigenfunctions of Δ (stationary solutions), due to Bourgain-Jakobson 1997 using Fourier series expansion of solutions.

For $V = 0$, Macià had given a microlocal proof in dimension $d = 2$.

Burq (2013) shows that the absolute continuity result for $V = 0$ implies the result for $V \in L^1_{loc}(\mathbb{R}_t, \mathcal{L}(L^2(\mathbb{T}^d)))$.

Reference to other work :
The observability result was proven by Jaffard and Haraux (90), for $V = 0$ (using Fourier expansions).

Burq-Zworski give a microlocal proof in $d = 2$ ($V = 0$ 2003, $V \neq 0$, continuous, time-independent, 2011).

Our observability result holds in arbitrary dimension, for V continuous, or Riemann integrable. Bourgain-Burq-Zworski (2012) manage to lower the regularity to $V \in L^2$, in dimension $d = 2$.

Extension

This work has been extended by A-Fermanian-Macià to more general (semiclassical) equations on $\mathbb{T}^d$,

$$i\hbar \frac{\partial u}{\partial t} = (H(\hbar D_x) + \hbar^2 V)u$$

where we impose the (necessary) condition $d^2 H > 0$.

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where we impose the (necessary) condition $d^2 H > 0$.

Using locally the Arnold-Liouville theorem, this also implies results for more general, non-degenerate, completely integrable systems.

The disc

The Schrödinger equation

$$(S) \qquad i\frac{\partial u}{\partial t} = \left(\frac{\Delta}{2} + V\right) u$$

$$u_{\rceil t=0} = u^0$$

in the disc $\mathbb{D} = \{(x, y), x^2 + y^2 < 1\}$, with Dirichlet boundary condition.

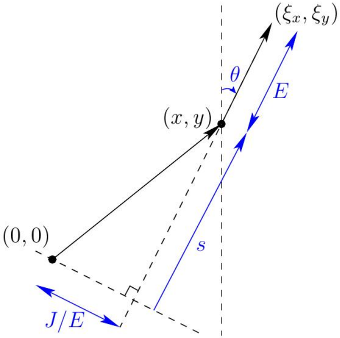
The classical dynamics is the billiard flow :

$$\phi^t : (x, \xi) \mapsto (x + t\xi, \xi)$$

as long as the trajectory stays inside the disc, and reflection on the boundary.

A nice set of coordinates for the dynamics is
$\Phi : (s, \theta, E, J) \longrightarrow (x, y, \xi_x, \xi_y)$ where

$$\begin{cases} x = \frac{J}{E}\cos\theta - s\sin\theta, \\ y = \frac{J}{E}\sin\theta + s\cos\theta, \\ \xi_x = -E\sin\theta, \\ \xi_y = E\cos\theta. \end{cases}$$



In these coordinates, the flow ϕ^t reads (away from reflections)

$$(s, \theta, E, J) \mapsto (s + tE, \theta, E, J)$$

and the rotation of angle τ around the origin $\mathbb{R}^\tau$ (generated by J) reads

$$(s, \theta, E, J) \mapsto (s, \theta + \tau, E, J)$$

The angle of the trajectory with the boundary is

$$\alpha = -\arcsin\left(\frac{J}{E}\right)$$

. The equation of the boundary is

$$s = \pm\cos\alpha = \pm\sqrt{1 - \frac{J^2}{E^2}}.$$

The reflection on the boundary reads

$$(\cos\alpha, \theta, E, J) \mapsto (-\cos\alpha, \theta + 2\alpha, E, J).$$

E and J (and α) are conserved.

A-Macià-Léautaud 2014

Take again (u_n^0) a sequence of initial conditions, normalized in $L^2(\mathbb{D})$. We extend is by 0 outside the disc and call $u_n(x,t)$ the associated solutions (with Dirichlet boundary condition)
Again we consider

$$W_{u_n}(a) = \langle u_n, a(x, D_x, t, D_t)u_n \rangle_{L^2(\mathbb{D}\times\mathbb{R})},$$

$a \in \mathcal{S}^0$.

We prove a structure theorem for the limits μ of the sequence W_{u_n}.
It has the following consequences :

Theorem

Let (u_n^0) be a sequence in $L^2(\mathbb{D})$, such that $\|u_n^0\|_{L^2(\mathbb{D})} = 1$ for all n. Consider the sequence of positive Radon measures ν_n on $\overline{\mathbb{D}} \times \mathbb{R}$, defined by

$$\nu_n(dx, dt) = |u_n(x, t)|^2 dxdt.$$

Let ν be any weak-$*$ limit of the sequence (ν_n) : then $\nu(dx, dt) = \nu_t(dx)dt$ where, for almost every t, ν_t is a probability measure on $\overline{\mathbb{D}}$, and $\nu_t\rceil_{\mathbb{D}}$ is absolutely continuous.

Unique continuation, observability

Theorem

Let $\Omega \subset \overline{\mathbb{D}}$ be an open set such that $\Omega \cap \partial\mathbb{D} \neq \emptyset$ and fix any $T > 0$. Assume one the the following statements holds :

- *the potential $V \in C^\infty([0, T] \times \overline{\mathbb{D}}; \mathbb{R})$ and the open set Ω satisfy $(\mathrm{UCP}_{V,\Omega})$,*
- *the potential $V \in C^0(\overline{\mathbb{D}}; \mathbb{R})$ does not depend on t.*

Then there exists $C = C(T, \Omega) > 0$ such that :

$$\left\|u^0\right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \|u(t)\|^2_{L^2(\Omega)}\, dt, \tag{4.1}$$

for every initial datum $u^0 \in L^2(\mathbb{D})$.

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$$\left\| u^0 \right\|^2_{L^2(\mathbb{D})} \leq C \int_0^T \|u(t)\|^2_{L^2(\Omega)}\, dt, \qquad (4.1)$$

for every initial datum $u^0 \in L^2(\mathbb{D})$.

$$u^0 \in L^2(\mathbb{D}), \quad u(t)\rceil_{(0,T)\times\Omega} = 0 \Longrightarrow u^0 = 0. \qquad (\mathrm{UCP}_{V,\Omega})$$

Boundary observability

Theorem

Let Γ *be any nonempty subset of* $\partial\mathbb{D}$ *and fix any* $T > 0$. *Suppose one of the following holds :*

- $V \in C^\infty([0,T] \times \overline{\mathbb{D}})$ *and* Γ *satisfy* $(\mathrm{UCP}_{V,\Gamma})$,
- $V \in C^0(\overline{\mathbb{D}})$ *does not depend on* t.

Then there exists $C = C(T,\Gamma) > 0$ *such that :*

$$\left\| u^0 \right\|^2_{H^1(\mathbb{D})} \leq C \int_0^T \|\partial_n u(t)\|^2_{L^2(\Gamma)}\, dt,$$

for every initial datum $u^0 \in H^1_0(\mathbb{D})$.

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$$\left\|u^0\right\|^2_{H^1(\mathbb{D})} \leq C \int_0^T \|\partial_n u(t)\|^2_{L^2(\Gamma)}\, dt,$$

for every initial datum $u^0 \in H^1_0(\mathbb{D})$.

$$u^0 \in H^1_0(\mathbb{D}), \quad \partial_n u\rceil_{(0,T)\times\Gamma} = 0 \Longrightarrow u^0 = 0, \qquad (\mathrm{UCP}_{V,\Gamma})$$

(UCP$_{V,\Omega}$) and (UCP$_{V,\Gamma}$) are known to hold when V is analytic in (t, x) (Holmgren).