Toeplitz operators and asymptotic torsion

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À la mémoire de Louis Boutet de Monvel

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- Example:
  $X = S^1, F = \mathbf{C}, \nabla^F = dx\left(\frac{\partial}{\partial x} + \alpha\right), \alpha \in \mathbf{C} \setminus 2i\pi\mathbf{Z}$.

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- If $n = \dim X$ is odd, analytic torsion does not depend on the metrics g^{TX}, g^{F}.

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- If g^F flat, $T_{\mathrm{comb}}\left(K,F\right)$ does not depend on K, defines Reidemeister torsion.

The Cheeger-Müller theorem

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Theorem

(Cheeger-Müller) If g^F flat, analytic torsion = Reidemeister torsion.

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- As $T \to +\infty$, $\square_T^X = \square^X + T^2 \left|\nabla f\right|^2 + T \dots$.
- As $T \to +\infty$, proof localizes near critical points of f.

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- If $F = \mathbf{R}, g^F = e^{-2Tf}\left|\,\right|^2$, $\square^X_T = -\Delta^{X,u} + T^2\left|\nabla f\right|^2 + \cdot$.

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- Spectral gap will be obtained by expressing the zero order term as a Toeplitz operator.

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- Then $F = H^0\left(\mathbf{P}^{F^*}, L\right)$.
- For $p \in \mathbf{N}$, $S^p F = H^0\left(\mathbf{P}^{F^*}, L^p\right)$ dim. $\binom{p+n-1}{n-1}$.

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- Family of flat vector bundles $F_p|_{p \in \mathbf{N}}$ on X.

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- For $p \in \mathbf{N}$ large enough, $H^i\left(N, L^p\right) = 0$ for $i > 0$.
- g^{F_p} fibrewise L_2 metric on $F_p = H^0\left(N, L^p\right)$.

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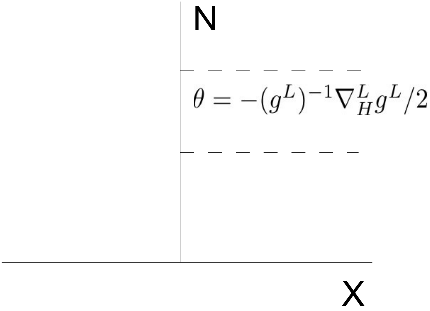

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  $\omega^{F,2}(U,V) = \left[\omega^F(U), \omega^F(V)\right]$.

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- $\ldots$ and $\frac{1}{4p}\omega^{F_p,2} = T_{\theta^{*2},p} + \mathcal{O}(1/p)$.
- θ^{*2} Poisson bracket $\theta^{*2}(U,V) = \{\theta(U), \theta(V)\}$.

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- ...and local on fibre N (Toeplitz operators).

The behaviour of analytic torsion as $p \to +\infty$

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Theorem

(B., Ma, Zhang) As $p \to +\infty$, if $n = \dim N$,

$$p^{-n-1} T_{\mathrm{an},p} = W + \mathcal{O}\left(1/p\right).$$

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- Formula for W related to B-Zhang formula

$$T_{\mathrm{comb}} - T_{\mathrm{an}} = \int_X \frac{1}{2} \mathrm{Tr}\left[\omega^F\right] (\nabla f)^* \psi.$$

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- P (fibrewise) orthogonal projection on $\ker H$.

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- $\left(\lambda - L_b^X\right)^{-1} \to P\left(\lambda + \frac{1}{2}\Delta^X\right)^{-1} P$ by collapsing of $\mathcal{X}$ on X.

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Theorem

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The case of S^1

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- Poisson formula for the heat kernel can be proved using the hypoelliptic interpolation.

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- If X Riemann surface, L_b^X acts on a manifold of dimension $5 = 2 + 3$.

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$$L_b^X = \frac{1}{2}\left|\left[Y^N, Y^{TX}\right]\right|^2 + \frac{1}{2b^2}\left(-\Delta^{TX\oplus N} + |Y|^2 - n\right) + \frac{N^{\Lambda^{\cdot}(T^*X \oplus N^*)}}{b^2}$$

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L_b^X is a deformation of $-\frac{1}{2}\left(\Delta^X + c\right)$.

The preservation of the trace

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Theorem

B. For any $t > 0, b > 0$

$$\mathrm{Tr}\left[\exp\left(t(\Delta^X + c\right)/2\right] = \mathrm{Tr_s}\left[\exp\left(-t\mathcal{L}_b^X\right)\right].$$

The limit as $b \to +\infty$

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- As $b \to +\infty$, concentration on closed geodesics.

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$$\mathrm{Tr}^{[\gamma]}\left[\exp\left(t\left(\Delta^X+c\right)/2\right)\right]=\frac{\exp\left(-\left|a\right|^2/2t\right)}{\left(2\pi t\right)^{p/2}}$$

$$\int_{\mathfrak{k}(\gamma)} J_\gamma\left(Y_0^{\mathfrak{k}}\right)\mathrm{Tr}^E\left[\rho^E\left(k^{-1}\right)\exp\left(-i\rho^E\left(Y_0^{\mathfrak{k}}\right)\right)\right]$$

$$\exp\left(-\left|Y_0^{\mathfrak{k}}\right|^2/2t\right)\frac{dY_0^{\mathfrak{k}}}{\left(2\pi t\right)^{q/2}}.$$

Semisimple orbital integrals

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- Like fixed point formulas by Atiyah-Bott

$$L\left(g\right) = \int_{X_g} \widehat{A}_g\left(TX\right)\mathrm{ch}_g\left(E\right).$$

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$$\left.\frac{\det\left(1-\exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{k}_0^\perp(\gamma)}}{\det\left(1-\exp\left(-i\mathrm{ad}\left(Y_0^{\mathfrak{k}}\right)\right)\mathrm{Ad}\left(k^{-1}\right)\right)|_{\mathfrak{p}_0^\perp(\gamma)}}\right]^{1/2}.$$

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- …which is also the symbol of Fourier transform.
- Hypoelliptic Laplacian gives dynamic interpretation of Fourier transform.

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- How does the hypoelliptic heat flow propagate ?

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- By B, Lebeau, as $b \to 0$, $q_{b,t}\left((x,Y),(x',Y')\right) \to \pi^{-n/2} p_t\left(x,x'\right) \exp\left(-\frac{1}{2}\left(\left|Y\right|^2 + \left|Y'\right|^2\right)\right)$.

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- $r_{b,t}\left(\left(x,Y\right),x'\right) = \int_{TX} q_{b,t}\left(\left(x,Y\right),\left(x',Y'\right)\right)dY'$.

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- As $b \to 0$, in x', $r_{b,t}$ approximates solution of hyperbolic wave equation with propagation speed $1/b$.
- The heat flow $\exp\left(-tL_b^X\right)$ projects to an "intelligent" wave programmed to look for closed geodesics as $b \to +\infty$.

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