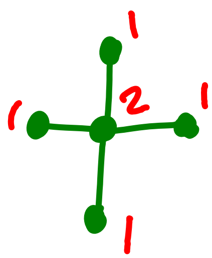
Non-Perturbative symplectic manifolds

P. Boalch (CNRS & Orsay)

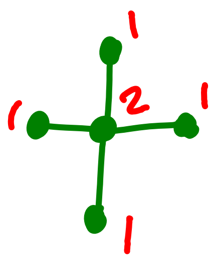
- Aim: Study symplectic/hyperkähler moduli spaces of connections on curves by viewing them as multiplicative/non-perturbative versions of simpler symplectic/hyperkähler manifolds

- **Aim:** Study symplectic / hyperkähler moduli spaces of connections on curves by viewing them as multiplicative / non-perturbative versions of simpler symplectic / hyperkähler manifolds

Eg 

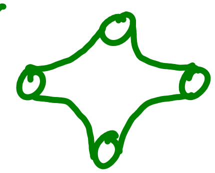
$\Longrightarrow$ D_4 ALE space

• Aim: Study symplectic / hyperkähler moduli spaces of connections on curves by viewing them as multiplicative / non-perturbative versions of simpler symplectic / hyperkähler manifolds

Eg 

$\Longrightarrow$ D_4 ALE space

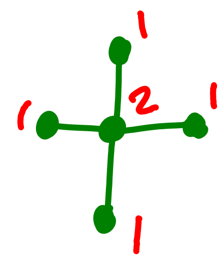
$\cap$

$\mathcal{M}($  $, GL_2)$

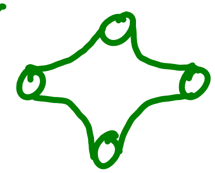
$\cong$

$$\mathcal{M}_{Betti} \cong \left\{ \begin{array}{l} xyz + x^2+y^2+z^2 \\ \quad = ax+by+cz+d \end{array} \right\} \subset \mathbb{C}^3$$

• Aim: Study symplectic / hyperkähler moduli spaces of connections on curves by viewing them as multiplicative / non-perturbative versions of simpler symplectic / hyperkähler manifolds

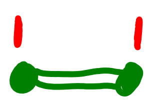
Eg  $\Rightarrow$ D_4 ALE space

$\cap$

$\mathcal{M}\Big($  $, \; GL_2\Big)$

$\cong$

$$\mathcal{M}_{Betti} \cong \left\{ \begin{array}{r} xyz + x^2 + y^2 + z^2 \\ = ax + by + cz + d \end{array} \right\} \subset \mathbb{C}^3$$

What about  $\Rightarrow$ Eguchi-Hanson manifold $\subset$?

$T^*\mathbb{P}^1 \sim \mathcal{O} \subset sl_2(\mathbb{C})$

- Motivation : classify nonlinear differential equations
  - algebraic integrable systems
  - isomonodromy equations

- Motivation : classify nonlinear differential equations
  - algebraic integrable systems
  - isomonodromy equations
- such moduli spaces of connections have complete hyperkähler metrics $\left\{\begin{array}{l}\text{Hitchin '87}\\ \text{Biquard - B. '04}\end{array}\right.$

- **Motivation**: classify nonlinear differential equations
  - algebraic integrable systems
  - isomonodromy equations

- such moduli spaces of connections have complete hyperkähler metrics $\begin{cases}\text{Hitchin '87} \\ \text{Biquard – B. '04}\end{cases}$

Big picture

$$\begin{array}{ccccll}
 & & & & \mathcal{M}_{Dol} & \text{mero. Higgs bundles} \\
 & & & \nearrow & & \text{meromorphic Hitchin integrable systems} \\
 & & & & \wr\wr\; \text{wild nonabelian Hodge} & \\
\Sigma & \Rrightarrow & \mathcal{M}(\Sigma) & \rightarrow & \mathcal{M}_{DR} & \text{mero. connections} \\
 & & & & \wr\wr\; \text{Riemann–Hilbert–Birkhoff} & \\
 & & & \searrow & \mathcal{M}_{Betti} & \text{wild character varieties}
\end{array}$$

Σ: wild Riemann surface / irregular curve (+ weighted conjugacy class)

$\mathcal{M}(\Sigma)$: Hyperkähler manifold

(See e.g. survey arXiv:1203) – No known new examples beyond curves

Spaces from graphs/quivers

$\Gamma = \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ \qquad I = \{\text{nodes}(\Gamma)\}$

Spaces from graphs/quivers

$$\Gamma = \overset{V_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\overset{V_2}{\circ} \qquad I = \{\text{nodes}(\Gamma)\}$$

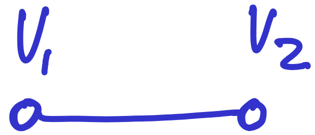
Spaces from graphs/quivers

$$\Gamma = \overset{V_1}{\circ}\!\!-\!\!-\!\!-\!\!-\!\!\overset{V_2}{\circ} \qquad I = \{\text{nodes}(\Gamma)\}$$

$$V = V_1 \oplus V_2$$

(I graded complex vector space)

Spaces from graphs/quivers

$\Gamma =$ 

$I = \{\text{nodes}(\Gamma)\}$

$V = V_1 \oplus V_2$ (I graded complex vector space)

$\text{Rep}(\Gamma, V) = \text{Hom}(V_1, V_2) \oplus \text{Hom}(V_2, V_1)$

Spaces from graphs/quivers

$$\Gamma = \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, no head] \arrow[r, bend left, "a"] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd} \qquad I = \{\text{nodes}(\Gamma)\}$$

$$V = V_1 \oplus V_2 \qquad (I \text{ graded complex vector space})$$

$$\mathrm{Rep}(\Gamma, V) = \underset{a}{\mathrm{Hom}(V_1, V_2)} \oplus \underset{b}{\mathrm{Hom}(V_2, V_1)}$$

Spaces from graphs/quivers

$$\Gamma = \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, no head] \arrow[r, bend left, "a"] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd} \qquad I = \{\text{nodes}(\Gamma)\}$$

$$V = V_1 \oplus V_2$$ (I graded complex vector space)

$$\mathrm{Rep}(\Gamma, V) = \underset{a}{\mathrm{Hom}(V_1, V_2)} \oplus \underset{b}{\mathrm{Hom}(V_2, V_1)}$$

$$\cong T^* \mathrm{Hom}(V_1, V_2)$$ (symplectic)

Spaces from graphs/quivers

$$\Gamma = \quad \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, dash] \arrow[r, bend left, "a"] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd} \qquad I = \{\text{nodes}(\Gamma)\}$$

$$V = V_1 \oplus V_2 \qquad (I \text{ graded complex vector space})$$

$$\mathrm{Rep}(\Gamma, V) = \underset{a}{\mathrm{Hom}(V_1, V_2)} \oplus \underset{b}{\mathrm{Hom}(V_2, V_1)}$$

$$\cong T^* \mathrm{Hom}(V_1, V_2) \qquad (\text{symplectic})$$

$H := GL(V_1) \times GL(V_2)$ acts on $\mathrm{Rep}(\Gamma, V)$

with moment map $\mu(a,b) = (ab, -ba)$

Spaces from graphs/quivers

$$\Gamma = \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, no head] \arrow[r, bend left, "a"] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd} \qquad I = \{ \text{nodes}(\Gamma) \}$$

$$V = V_1 \oplus V_2 \qquad (I \text{ graded complex vector space})$$

$$\mathrm{Rep}(\Gamma, V) = \underset{a}{\mathrm{Hom}(V_1, V_2)} \oplus \underset{b}{\mathrm{Hom}(V_2, V_1)}$$

$$\cong T^* \mathrm{Hom}(V_1, V_2) \qquad (\text{symplectic})$$

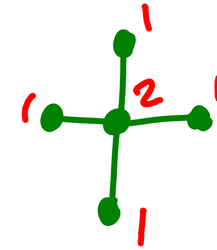
$H := GL(V_1) \times GL(V_2)$ acts on $\mathrm{Rep}(\Gamma, V)$

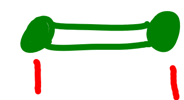
with moment map $\mu(a,b) = (ab, -ba)$

Additive/Nakajima quiver variety : $\quad \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H = \mu^{-1}(\lambda)/H \qquad (\lambda \in \mathbb{C}^I \subset \mathrm{Lie}(H)^*)$

Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then $\mathrm{Rep}(\Gamma, V) /\!\!/_\lambda H$ is cx dim^n 2





$$\mathrm{Rep}(\Gamma, V) = \underset{a}{\mathrm{Hom}(V_1, V_2)} \oplus \underset{b}{\mathrm{Hom}(V_2, V_1)}$$

$$\cong T^* \mathrm{Hom}(V_1, V_2) \qquad \text{(symplectic)}$$

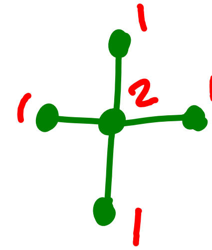
$$H := GL(V_1) \times GL(V_2) \quad \text{acts on } \mathrm{Rep}(\Gamma, V)$$

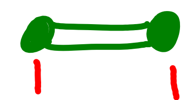
with moment map $\mu(a,b) = (ab, -ba)$

Additive/Nakajima quiver variety: $\mathrm{Rep}(\Gamma, V) /\!\!/_\lambda H = \mu^{-1}(\lambda)/H \quad \left(\lambda \in \mathbb{C}^I \subset \mathrm{Lie}(H)^*\right)$

Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then $\mathrm{Rep}(\Gamma, V) /\!/_\lambda H$ is cx dim^n 2





Multiplicative version

$$\Gamma = \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, bend left, "a"] \arrow[r, no head] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd}$$

$$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$$

$$\cap$$

$$\mathrm{Rep}(\Gamma, V)$$

"invertible representations"

Séminaire E.N.S. (1979-1980)

Partie IV

Appendice II à l'exposé n°3

$\mathcal{D}$-MODULES HOLONÔMES RÉGULIERS EN UNE VARIABLE

par L. BOUTET de MONVEL

1. Soit X une courbe complexe (lisse). On a vu (exposé n° 3) que les équations différentielles à points singuliers réguliers sur X sont complètement classifiées par leur monodromie. De façon plus précise : soit E un fibré holomorphe sur X, $a_1 \ldots a_n \ldots$ une famille discrète de points de X, $\nabla : E \longrightarrow E \otimes \Omega^1$ une équation différentielle (connexion) singulière aux a_j. Les solutions de l'équation $\nabla f = 0$ au voisinage de x_o correspondent bijectivement à leurs valeurs en x_o, et se prolongent au revêtement universel de $X \setminus \{a_1 \ldots a_n \ldots\}$, ce qui définit une action (monodromie) du groupe fondamental $\pi_1(X-\{a_1 \ldots a_n\}, x_o)$ dans l'espace des solutions. Alors

Progress in Mathematics

Edited by
J. Coates and
S. Helgason

Mathématique et Physique

Séminaire de l'Ecole Normale Supérieure 1979-1982

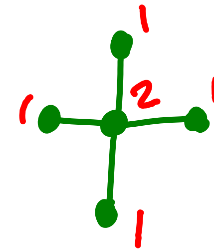
Louis Boutet de Monvel
Adrien Douady
Jean-Louis Verdier, editors

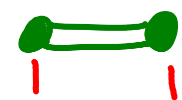
Birkhäuser

Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph, $\dim V_i \sim$ minimal null root then

$\mathrm{Rep}(\Gamma, V) /\!/_{\lambda} H$ is cx dim^n 2





Multiplicative version

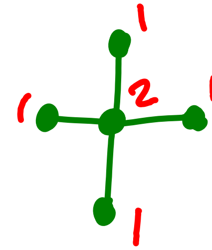
$$\Gamma = \begin{tikzcd} \overset{V_1}{\circ} \arrow[r, bend left, "a"] \arrow[r, dash] & \overset{V_2}{\circ} \arrow[l, bend left, "b"] \end{tikzcd}$$

$$\begin{array}{c} \mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \} \\ \cap \\ \mathrm{Rep}(\Gamma, V) \end{array}$$

"invertible representations"

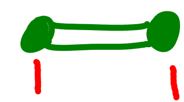
Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph,



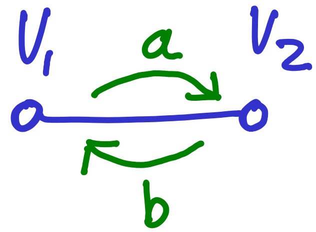
$\dim V_i \sim$ minimal null root then

$\mathrm{Rep}(\Gamma, V) /\!/_{\lambda} H$ is cx dim^n 2



Multiplicative version

$\Gamma =$



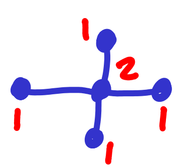
$$\mathrm{Rep}^*(\Gamma, V) = \{(a,b) \mid 1+ab \text{ invertible}\}$$

$$\cap$$

$$\mathrm{Rep}(\Gamma, V)$$

"invertible representations"

Thm (VandenBergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = \left(1+ab, (1+ba)^{-1}\right) \in H$

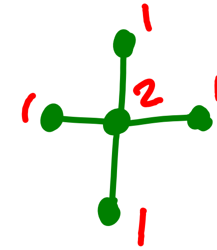
E.g. Mult. Quiver Var. () $\cong \{xyz + x^2+y^2+z^2 = ax+by+cz+d\}$

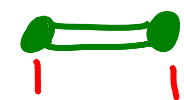
Spaces from graphs/quivers

Kronheimer '89: If Γ an affine ADE Dynkin graph,

$\dim V_i \sim$ minimal null root then

$\mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ is cx $\dim^n$ 2





Multiplicative version

$\mathcal{B}(V_1, V_2) :=$

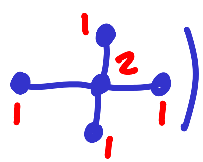
$$\Gamma = \begin{tikzcd} V_1 \arrow[r, bend left, "a"] & V_2 \arrow[l, bend left, "b"] \end{tikzcd}$$

$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$

$\cap$ "invertible representations"

$\mathrm{Rep}(\Gamma, V)$

Thm (Van den Bergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = (1+ab, (1+ba)^{-1}) \in H$

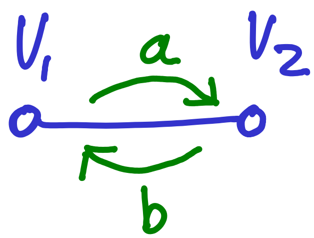
E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

Qn Suppose $\Gamma =$  or  etc

then what is $\mathrm{Rep}^*(\Gamma, V)$?

Multiplicative version

$\mathcal{B}(V_1, V_2) :=$

$\Gamma =$ 

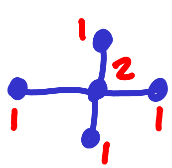
$$\mathrm{Rep}^*(\Gamma, V) = \{ (a,b) \mid 1+ab \text{ invertible} \}$$

$\cap$

$\mathrm{Rep}(\Gamma, V)$

"invertible representations"

Thm (Van den Bergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = (1+ab, (1+ba)^{-1}) \in H$

E.g. Mult. Quiver Var. () $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

SPECIMEN ALGORITHMI SINGVLARIS.

Auctore
L. EVLERO.

1.

Consideratio fractionum continuarum, quarum vsum vberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, vt singularem algorithmum requirat. Cum igitur summa Analyseos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus:

$$(a) = a$$
$$(a,b) = ab + 1$$
$$(a,b,c) = abc + c + a$$
$$(a,b,c,d) = abcd + cd + ad + ab + 1$$
$$(a,b,c,d,e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

ex

"Euler's continuant polynomials"

G. G. Stokes 1857

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]

IN a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account *.

These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient

How to define "multiplicative version"?

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

$X \in \mathfrak{g}$ $\Rightarrow$ $\exp(2\pi i X) \in G$

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

$$X \in \mathfrak{g} \quad \Longrightarrow \quad \exp(2\pi i X) \in G$$

$$\exp(2\pi i X) = \text{monodromy of } X \frac{dz}{z}$$

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

$$X \in \mathfrak{g} \quad \Longrightarrow \quad \exp(2\pi i X) \in G$$

$$\exp(2\pi i X) = \text{monodromy of } X \frac{dz}{z}$$

– can look at "monodromy" of many other connections

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

$X \in \mathfrak{g}$ $\Longrightarrow$ $\exp(2\pi i X) \in G$

$\qquad\qquad\qquad\qquad\qquad \|$

monodromy of $X \dfrac{dz}{z}$

- can look at "monodromy" of many other connections

$\left(\dfrac{A}{z} + \dfrac{B}{z-1}\right)dz \Longrightarrow$ all multizetas

(generating series is perturbative expansion about trivial connection of connection matrix $0 \leftrightarrow 1$)

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

$X \in \mathfrak{g}$ $\Longrightarrow$ $\exp(2\pi i X) \in G$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \|$

monodromy of $X \frac{dz}{z}$

- can look at "monodromy" of many other connections

$$\left(\frac{A}{z} + \frac{B}{z-1}\right)dz \Longrightarrow \text{all multizetas}$$

(generating series is perturbative expansion about trivial connection of connection matrix $0 \leftrightarrow 1$)

$$\left(\frac{A}{z^2} + \frac{B}{z}\right)dz \Longrightarrow \text{Poisson Lie group underlying } U_q\mathfrak{g}$$

$$\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\,\mathfrak{g} \xrightarrow[\text{quantize}]{} U\mathfrak{g}$$

(deform multiplication)

$$\begin{tikzcd}[column sep=huge, row sep=huge]
 & U_q\mathfrak{g} \\
\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\,\mathfrak{g} \arrow[r, rightsquigarrow, "\substack{\text{quantize} \\ \text{(deform multiplication)}}"'] & U\mathfrak{g} \arrow[u, rightsquigarrow, "\substack{\text{``quantize''} \\ \text{(deform comultiplication)}}"']
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathbb{C}[G^*] & & U_q\mathfrak{g} \\
\mathbb{C}[\mathfrak{g}^*] = \operatorname{Sym}\mathfrak{g} \arrow[u, rightsquigarrow, "\text{globalize}"] \arrow[rr, rightsquigarrow, "\substack{\text{quantize} \\ \text{(deform multiplication)}}"'] & & U\mathfrak{g} \arrow[u, rightsquigarrow, "\substack{\text{“quantize”} \\ \text{(deform comultiplication)}}"']
\end{tikzcd}
$$

$$\begin{tikzcd}
\mathbb{C}[G^*] \arrow[r, squiggly, "\text{quantize}"] & U_q\mathfrak{g} \\
\mathbb{C}[\mathfrak{g}^*] = \operatorname{Sym}\mathfrak{g} \arrow[u, squiggly, "\text{globalize}"] \arrow[r, squiggly, "{\text{quantize}}"', "{\text{(deform multiplication)}}"'] & U\mathfrak{g} \arrow[u, squiggly, "\text{``quantize''}"', "\text{(deform comultiplication)}"']
\end{tikzcd}$$

$\cong T \rtimes (U_+ \times U_-)$

$$
\begin{tikzcd}
\mathbb{C}[G^*] \arrow[rr, squiggly, "\text{quantize}"] & & U_q\mathfrak{g} \\
\mathbb{C}[\mathfrak{g}^*] = \operatorname{Sym}\mathfrak{g} \arrow[u, squiggly, "\text{globalize}"] \arrow[rr, squiggly, "\text{quantize}"'] & & U\mathfrak{g} \arrow[u, squiggly, "\text{``quantize''}"']
\end{tikzcd}
$$

("quantize": deform comultiplication)

(quantize: deform multiplication)

$$\cong T \ltimes (U_+ \times U_-)$$

$$\begin{tikzcd}
\mathbb{C}[G^*] \arrow[rr, squiggly, "\text{quantize}"] & & U_q\mathfrak{g} \\
\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\,\mathfrak{g} \arrow[u, squiggly, "\text{globalize}"] \arrow[rr, squiggly, "\text{quantize}"'] & & U\mathfrak{g} \arrow[u, squiggly, "\text{“quantize”}"']
\end{tikzcd}$$

(deform comultiplication)

(deform multiplication)

Thm (2001) G^* is the space of monodromy / Stokes data of connections $\left(\frac{A}{z^2} + \frac{B}{z}\right) dz \Big|_{\text{unit disc}}$ $\quad A \in \mathfrak{t}_{reg}$ fixed, $B \in \mathfrak{g} \cong \mathfrak{g}^*$

and the desired nonlinear Poisson structure appears this way

Cartoon

Cartoon

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

Cartoon

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

$$\Big\downarrow \mu^{-1}(0)/G$$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Cartoon

∞-d Ham^n geometry

e.g connections on C^∞ bundles / Riemann surfaces

$\cup$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

$\Big\downarrow \mu^{-1}(0)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Cartoon

∞-d Ham^n geometry

e.g connections on C^∞ bundles / Riemann surfaces

$\cup$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

$\mu^{-1}(0)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

$\mu^{-1}(0)/\mathcal{G}$

Multiplicative symplectic geometry

Betti spaces, character varieties

Cartoon

∞-d Ham^n geometry
e.g connections on C^∞ bundles / Riemann surfaces

$\cup$

$/\!/ \mathcal{G}_1$

Hamiltonian geometry
$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

quasi-Hamiltonian geometry
$\mathcal{C} \subset G$, $D = G \times G$

$\mu^{-1}(0)/G$

Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Multiplicative symplectic geometry
Betti spaces, character varieties

Cartoon

∞-d Ham^n geometry
e.g connections on C^∞ bundles / Riemann surfaces

$\cup$

$/\!/ \mathcal{G}_1$

Hamiltonian geometry
$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

quasi-Hamiltonian geometry
$\mathcal{C} \subset G$, $D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient $\mu^{-1}(1)/G$

Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Multiplicative symplectic geometry
Betti spaces, character varieties

Cartoon

∞-d Ham^n geometry

e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$

$/\!/ g_1$

Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

quasi-Hamiltonian geometry

$\mathcal{C} \subset G$, $D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient $\mu^{-1}(1)/G$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Multiplicative symplectic geometry

Betti spaces, character varieties

$$\left\{ d - \sum \frac{A_i}{z - a_i} dz \;\middle|\; A_i \in \mathcal{O}_i, \sum A_i = 0 \right\} / G$$

Cartoon

∞-d Ham^n geometry
e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$

Hamiltonian geometry
$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

$//\mathcal{G}_1$

quasi-Hamiltonian geometry
$\mathcal{C} \subset G$, $D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient $\quad \mu^{-1}(1)/G$

Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m // G$

$\mathcal{M}^*$

RH $\Longrightarrow$

Multiplicative symplectic geometry
Betti spaces, character varieties

$\mathcal{M}_B$

$$\begin{tikzcd}
& \infty\text{-d Ham}^n \text{ geometry} \arrow[dl, phantom, "\cup"] \arrow[dr, squiggly, "//\mathcal{G}_1"] & \\
\text{Hamiltonian geometry} \arrow[d, "\mu^{-1}(0)/G"] & & \text{quasi-Hamiltonian geometry} \arrow[d, "\mu^{-1}(1)/G", "\text{mult. sp. quotient}"'] \\
\text{Additive symplectic geometry } (\mathcal{M}^*) \arrow[rr, Rightarrow, "RH"] & & \text{Multiplicative symplectic geometry } (\mathcal{M}_B)
\end{tikzcd}$$

Cartoon

∞-d Ham^n geometry
e.g. connections on C^∞ bundles / Riemann surfaces

$\cup$

$/\!/ \mathcal{G}_1$

Hamiltonian geometry
$\mathcal{O} \subset \mathfrak{g}^*, \quad T^*G$

quasi-Hamiltonian geometry
$\mathcal{C} \subset G, \quad D = G \times G$

$\mu^{-1}(0)/G$

mult. sp. quotient $\mu^{-1}(1)/G$

Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

$\mathcal{M}^*$

$\xRightarrow{\text{RHB}}$

Multiplicative symplectic geometry
Betti spaces, wild character varieties

$\mathcal{M}_B$

Wild Character Varieties

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = \underbrace{\mathrm{Hom}(\pi_1(\Sigma), G)/G}_{\text{symplectic variety}}$

$$\begin{array}{c} \mathcal{M}_B \\ \wr\!\!\parallel \ \mathrm{RH} \\ \mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom} \end{array}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface
with marked points
$\underset{\sim}{a} = (a_1, \dots, a_m)$

$\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$ symplectic variety

$\cong$ RH

$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/\text{isom}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B^{tame} = \mathrm{Hom}(\pi_1(\Sigma^\circ), G)/G$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$\mathcal{M}_B^{tame} \overset{RH}{\cong} \mathcal{M}_{DR} = \left\{ \begin{array}{c} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with reg. sing.s} \end{array} \right\} \Big/ \text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson scheme (∞-type)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B$

$$\mathcal{M}_B \underset{RHB}{\cong}$$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma^\circ\} / \text{isom}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface
with marked points
$\underset{\sim}{a} = (a_1, \dots, a_m)$
and irregular types
$\underset{\sim}{Q} = Q_1, \dots, Q_m$

$\Longrightarrow$ $\mathcal{M}_B$

$\cong$ RHB

$\mathcal{M}_{DR} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \ldots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\cong$ RHB

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with irreg. types $\underset{\sim}{Q}$ $\} / \text{isom}$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$Q_i \in \underbrace{\mathfrak{t}_i}_{\text{cartan subalg.}} \subset \mathfrak{g}((z_i))$$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^{\circ} = \Sigma \setminus \underset{\sim}{a}$

$\mathcal{M}_B \; \overset{\text{RHB}}{\cong} \; \mathcal{M}_{DR}$

$\mathcal{M}_{DR} = \{ \text{Alg. connections on } G\text{-bundles on } \Sigma^{\circ} \text{ with irreg. types } \underset{\sim}{Q} \} / \text{isom}$

$\nabla \cong dQ_i + \Lambda_i \dfrac{dz_i}{z_i} + \text{holom.}$

Cartan subalg.

e.g. $\quad Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i)) \qquad \mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Rightarrow$ wild character variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \ldots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \ldots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$\mathcal{M}_B \overset{RHB}{\cong} \mathcal{M}_{DR} = \left\{ \begin{array}{l} \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \\ \text{with irreg. types } \underset{\sim}{Q} \end{array} \right\} \Big/ \text{isom}$$

$$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$$

Cartan subalg.

e.g. $\quad Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i)) \qquad \mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

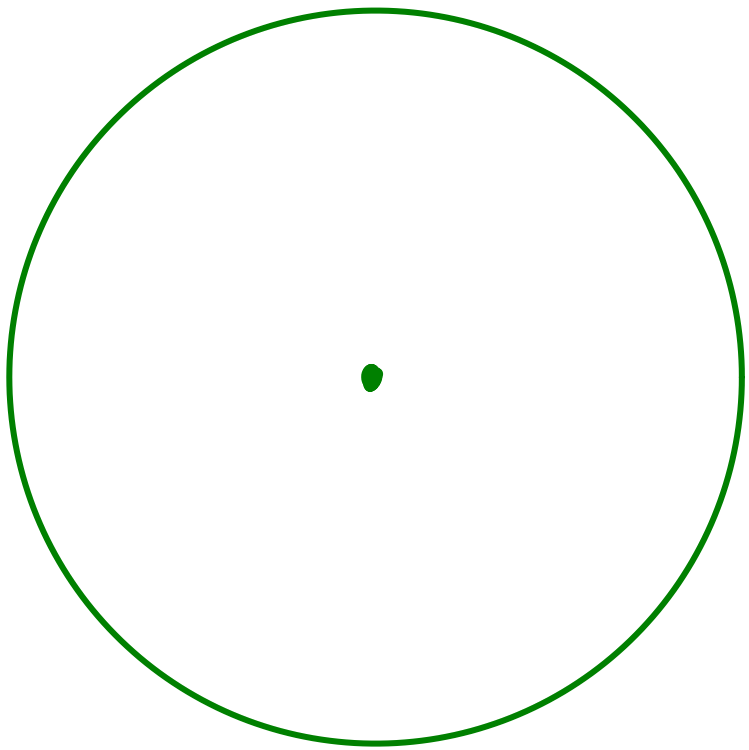
$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

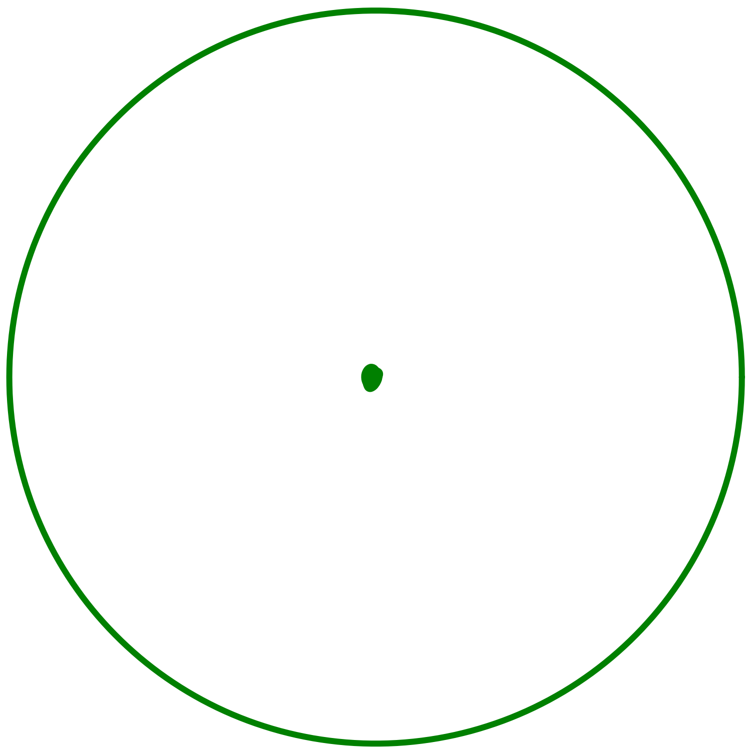


Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$



$Q \Rrightarrow$

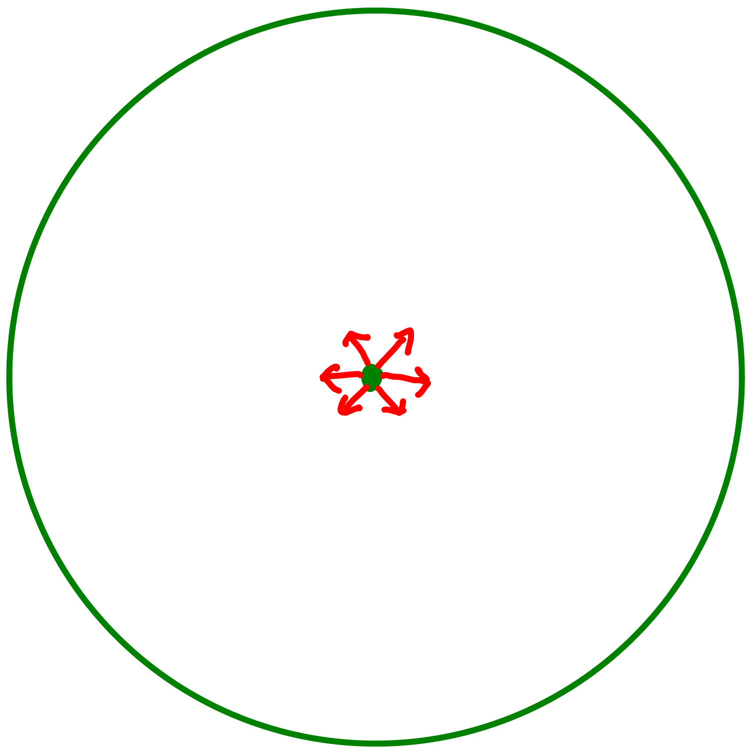
- centraliser group $H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
  $C_G(Q)$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. (Disc, 0, Q) $G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$



$Q \Rightarrow$

- centraliser group $C_G(Q)$ $H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$

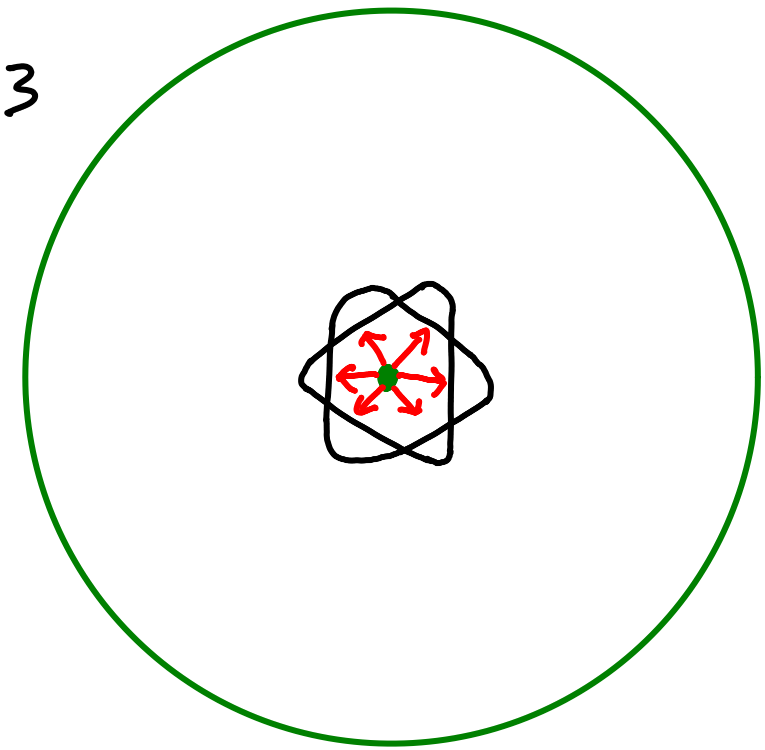
Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. (Disc, 0, Q) $\quad G = GL_2(\mathbb{C})$

$Q = A/z^k$, $\quad A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $\quad a \neq b$

$k=3$



$Q \Rrightarrow$

- centraliser group $C_G(Q)$ $\quad H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$

Solutions involve $\exp(Q)$

$Q = \mathrm{diag}(q_1, q_2)$

Stokes diagram: plot growth of $\exp(q_1)$, $\exp(q_2)$

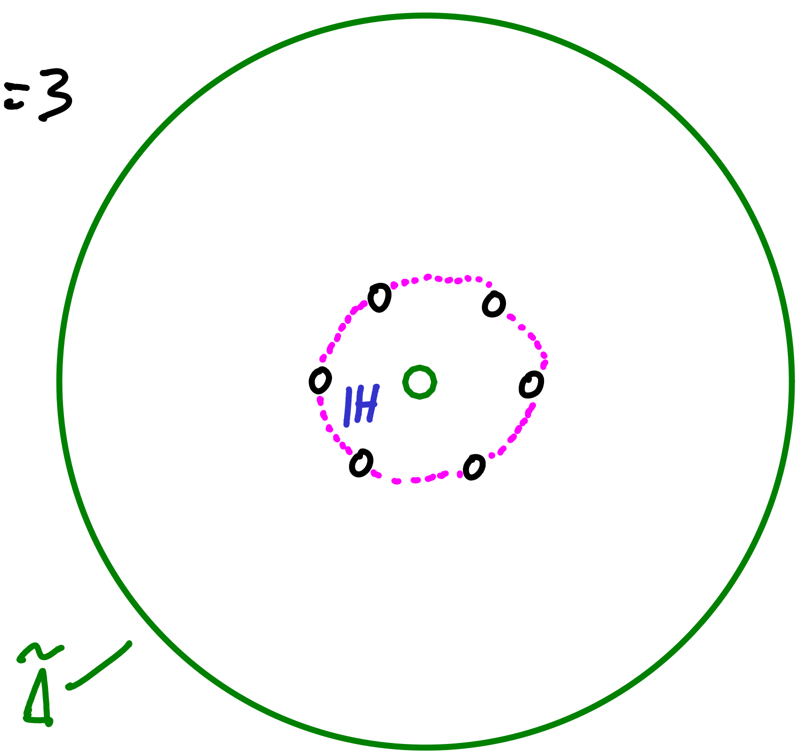
Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

$k=3$



o $e(d)$ extra punctures

IH halo/annulus

$Q \Rightarrow$

- centraliser group $C_G(Q)$ $\quad H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$

Solutions involve $\exp(Q)$

$$Q = \operatorname{diag}(q_1, q_2)$$

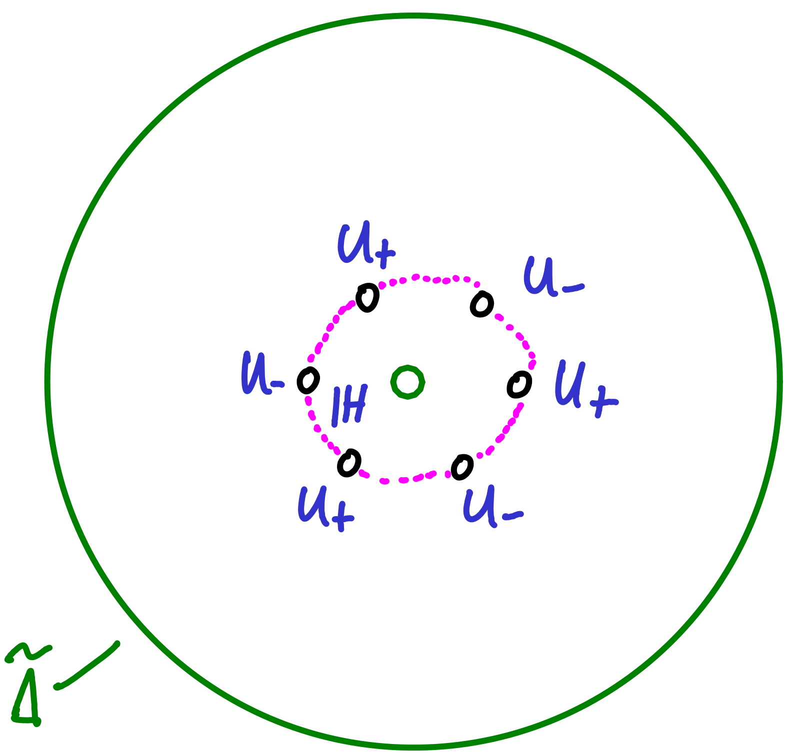
Stokes diagram: plot growth of $\exp(q_1)$, $\exp(q_2)$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. (Disc, 0, Q) $\qquad G = GL_2(\mathbb{C})$

$Q = A/z^k$, $\quad A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $\quad a \neq b$



$Q \Rightarrow$

- centraliser group $C_G(Q)$ $\quad H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
- singular directions $\mathbb{A}$
- Stokes groups $\mathbb{S}to_d \subset G \quad \forall d \in \mathbb{A}$

  $\cong U_+$ or U_- here

  $\begin{pmatrix} 1 & * \\ & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & \\ * & 1 \end{pmatrix}$

o $\quad e(d)$ extra punctures

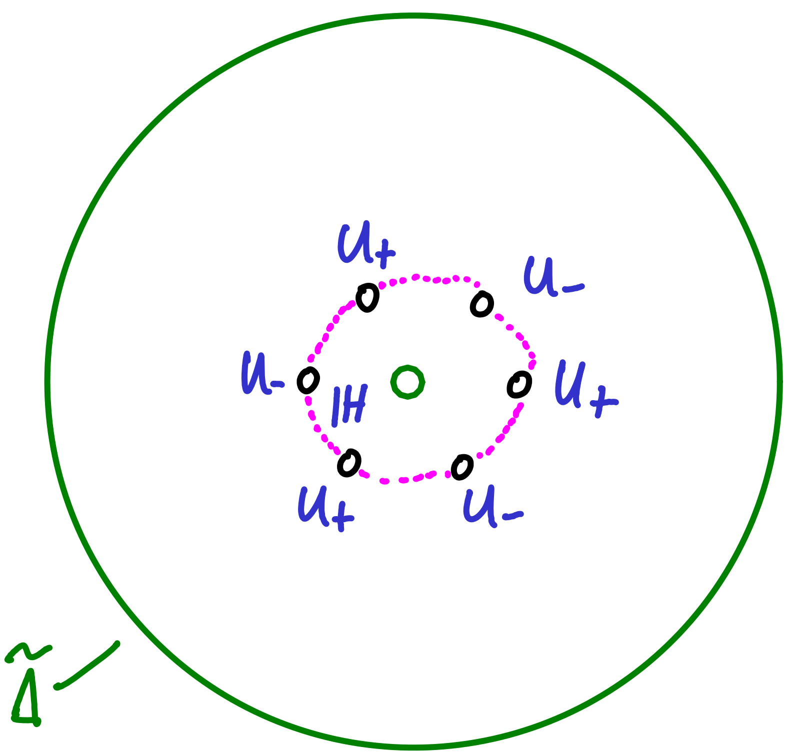
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



o $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

Stokes local system:

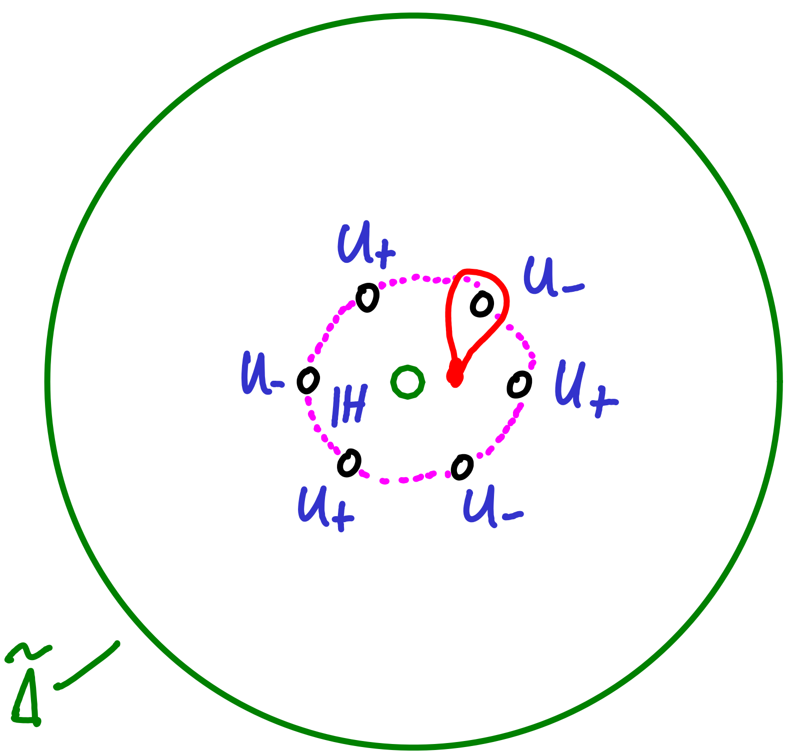
- G local system on $\tilde{\Delta}$
- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}\mathrm{to}_d$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



Stokes local system:

- G local system on $\tilde{\Delta}$
- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}to_d$

o $e(d)$ extra punctures

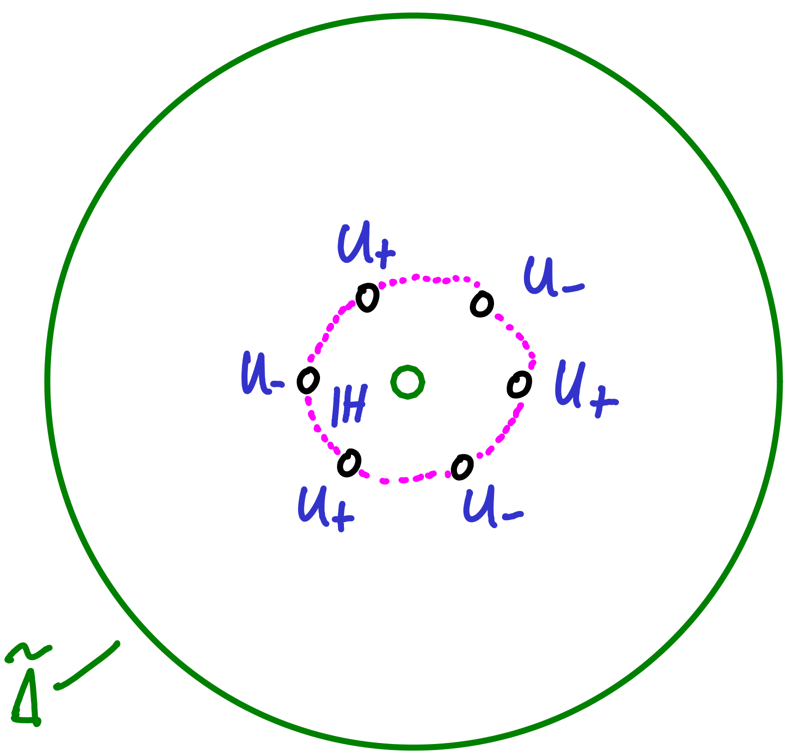
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. (Disc, 0, Q) $\qquad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



$\circ$ $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

Stokes local system:

- G local system on $\tilde{\Delta}$
- flat reduction to H in $\mathbb{H}$
- monodromy around $e(d)$ in $\mathbb{S}\mathrm{to}_d$

– Topological data that the multisummation approach to Stokes data gives

$$\left\{ \begin{array}{c} \text{Connections with} \\ \text{irreg. type } Q \end{array} \right\} \Longleftrightarrow \left\{ \begin{array}{c} \text{Stokes local} \\ \text{systems} \end{array} \right\}$$

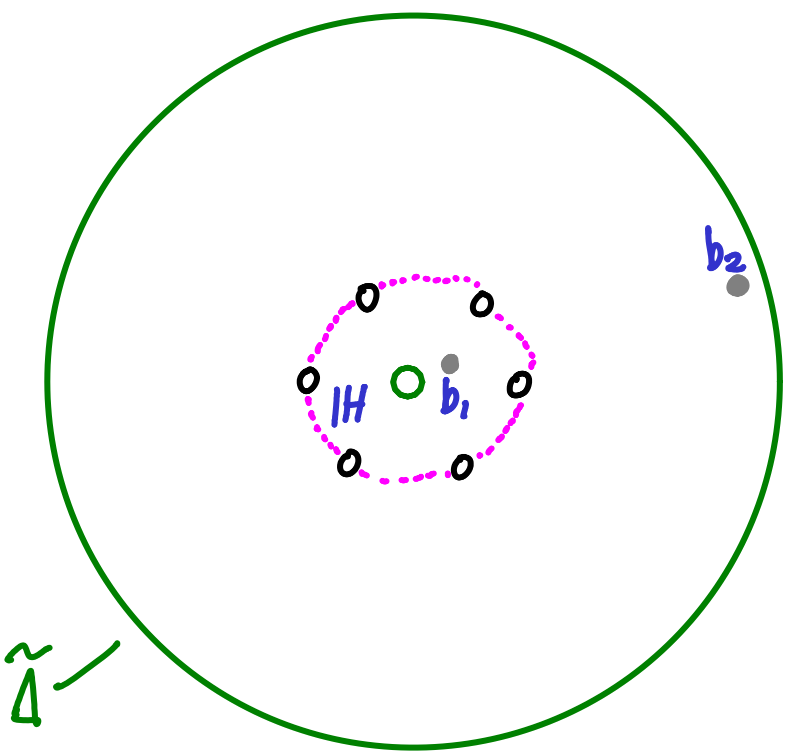
Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

basepoints b_1, b_2



o $\quad e(d)$ extra punctures

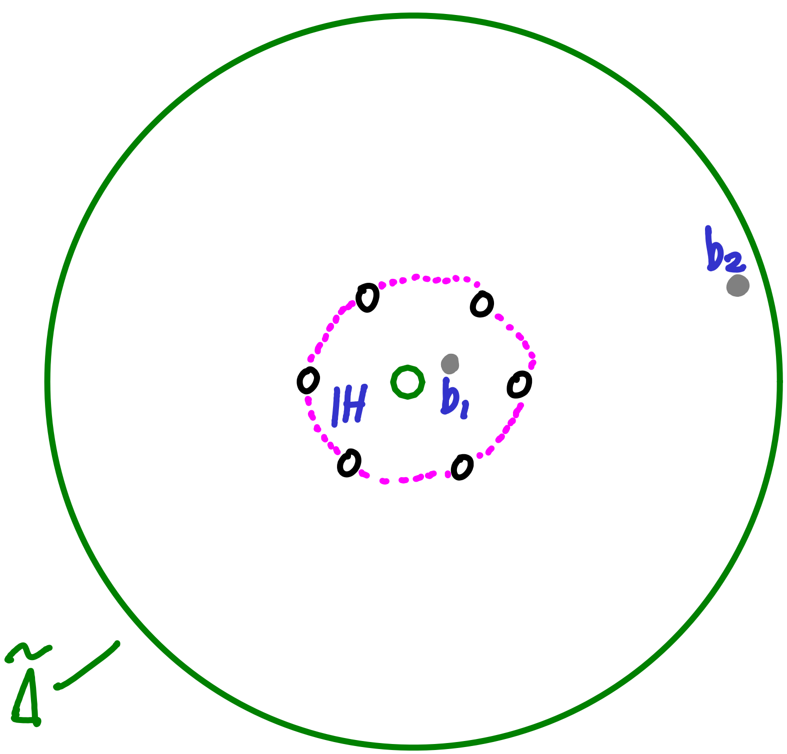
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$



basepoints b_1, b_2

$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

o $e(d)$ extra punctures

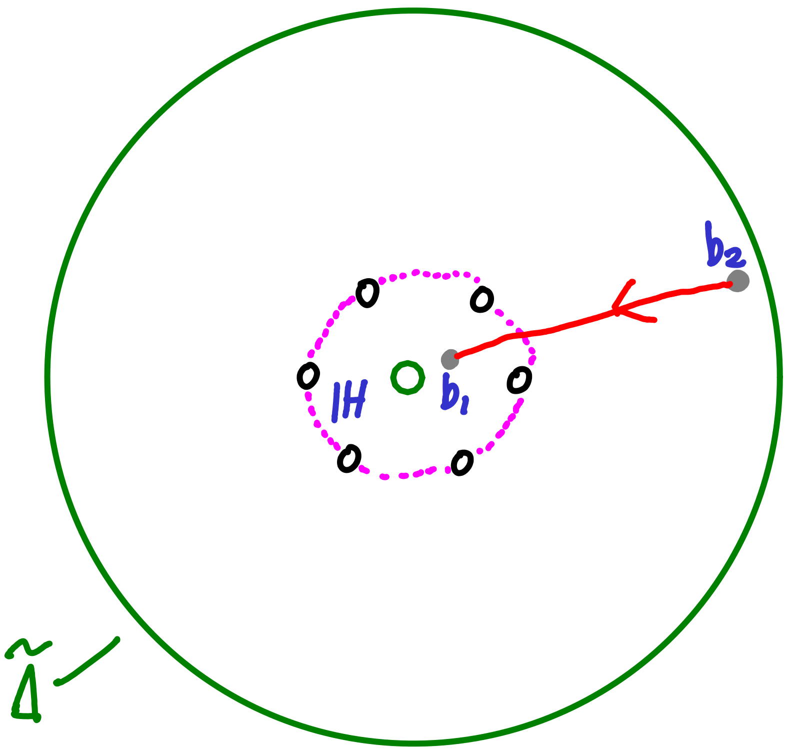
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. (Disc, 0, Q) $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



basepoints b_1, b_2

$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

o $e(d)$ extra punctures

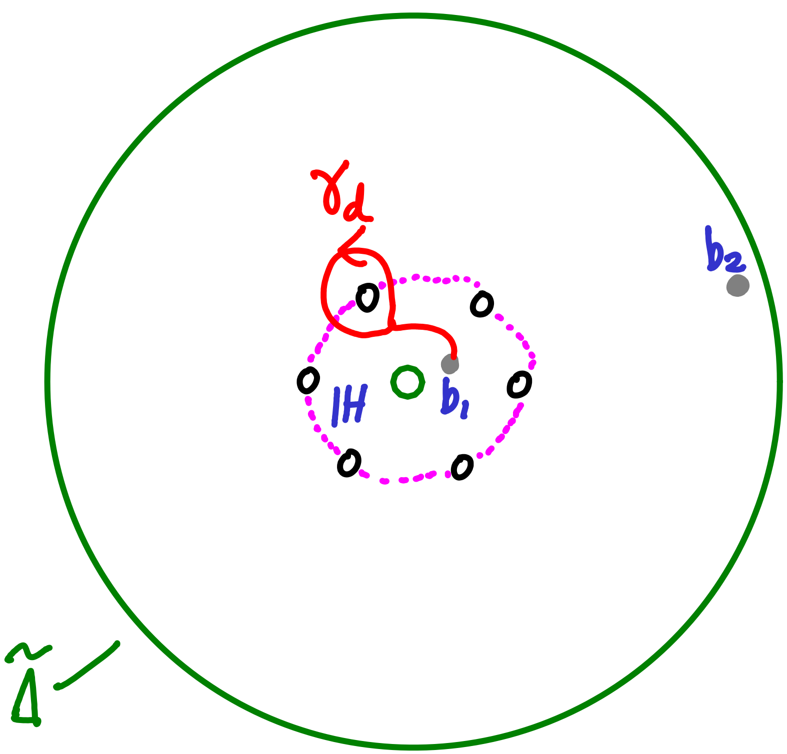
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. (Disc, 0, Q) $\qquad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



basepoints b_1, b_2

$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

o $\quad e(d)$ extra punctures

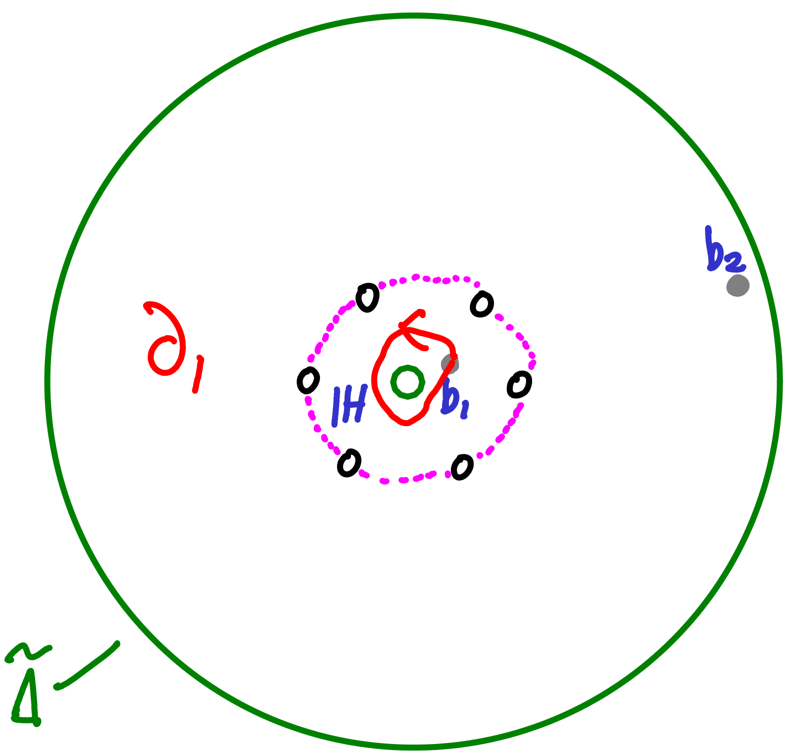
IH $\quad$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



basepoints b_1, b_2

$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

o $e(d)$ extra punctures

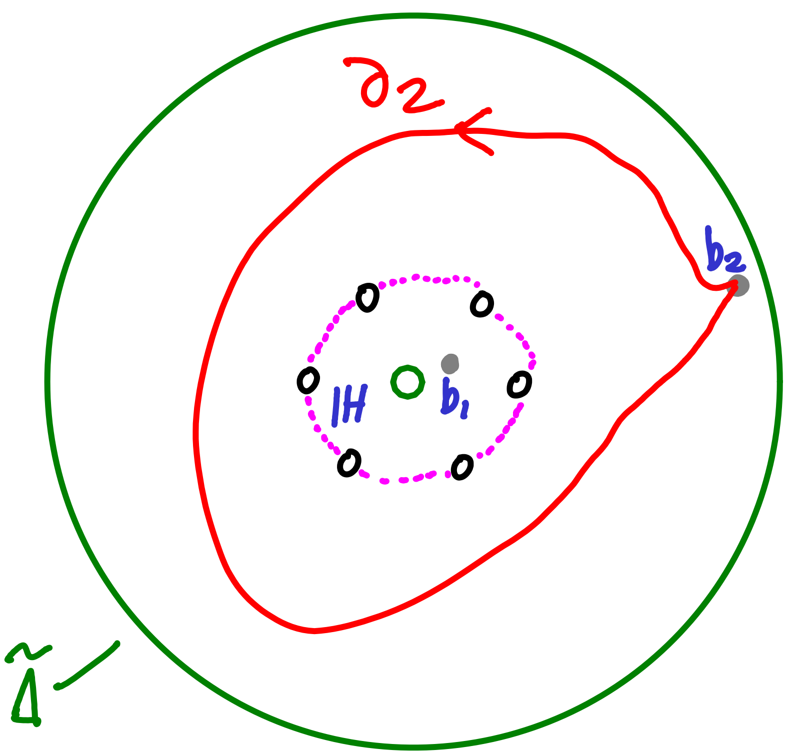
IH halo/annulus

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



basepoints b_1, b_2

$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

o $e(d)$ extra punctures

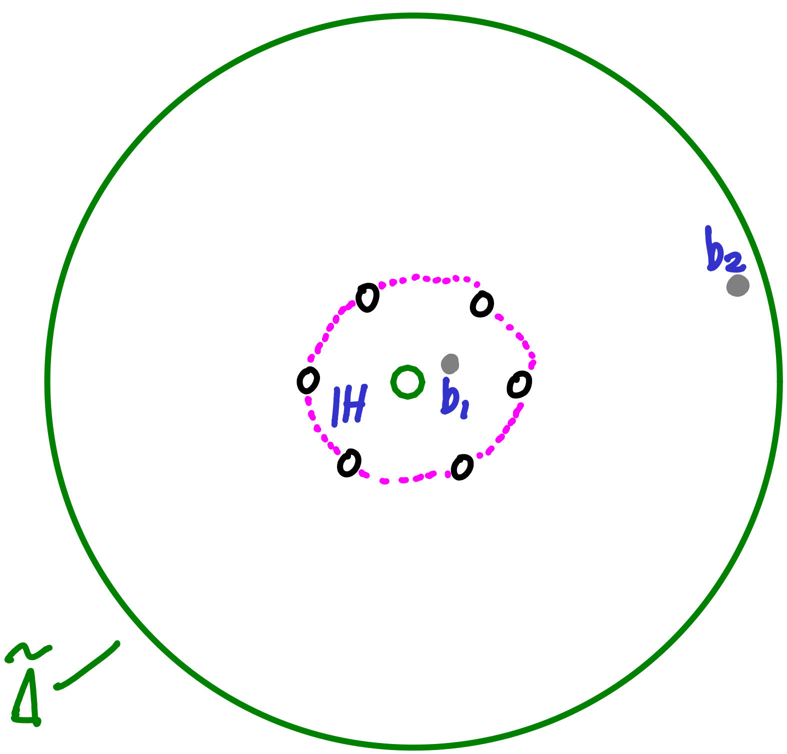
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$



basepoints b_1, b_2

$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

o $e(d)$ extra punctures

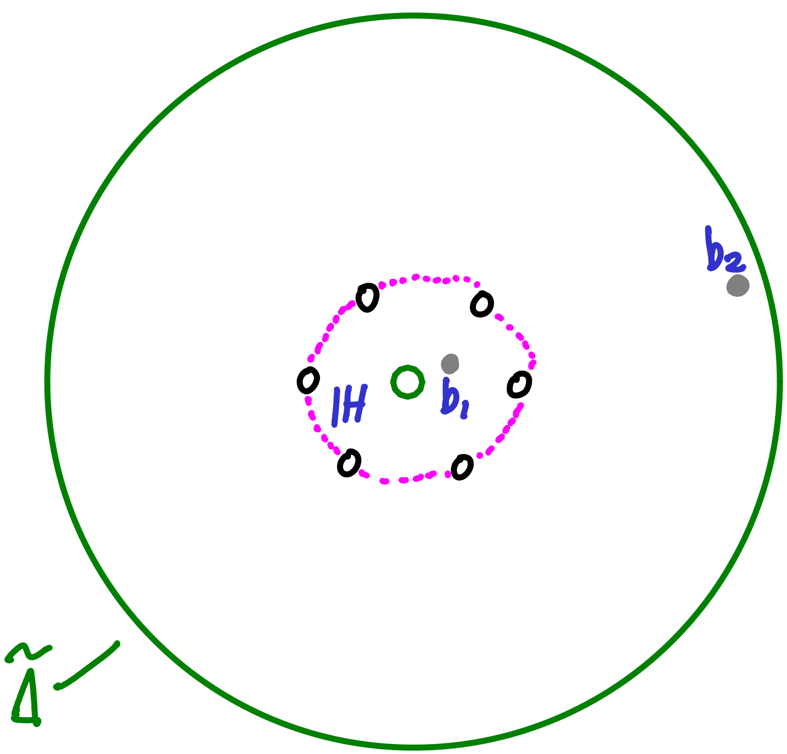
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$



basepoints b_1, b_2

$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

$\tilde{\mathcal{M}}_B = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$

$= \left\{ \rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(\partial_1) \in H \\ \rho(\gamma_d) \in \mathrm{Sto}_d \quad \forall d \in \mathbb{A} \end{array} \right\}$

o $e(d)$ extra punctures

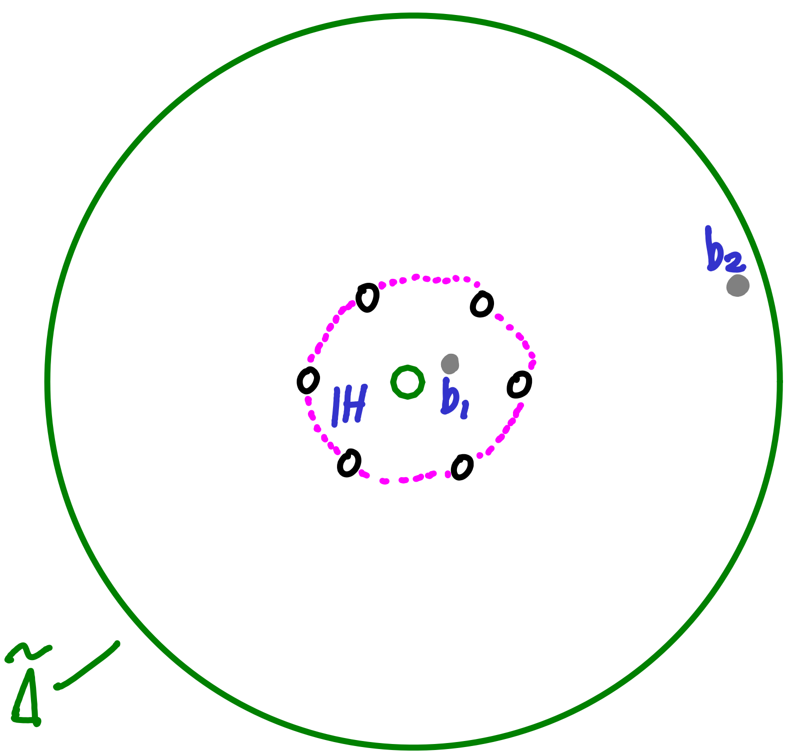
$\mathbb{H}$ halo/annulus

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$



$\circ$ $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

basepoints b_1, b_2

$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$

$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_g(\Pi, G)$$

$$= \left\{ \rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(\partial_1) \in H \\ \rho(\gamma_d) \in \mathrm{Sto}_d \quad \forall d \in \mathcal{A} \end{array} \right\}$$

Thm (arXiv 0203.*****)

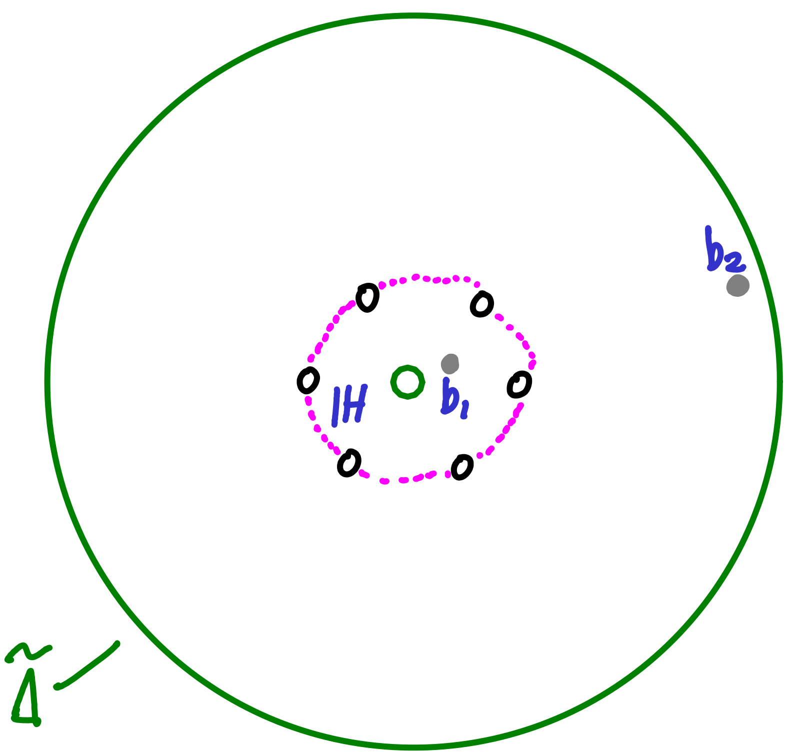
$\tilde{\mathcal{M}}_B$ is a quasi-Hamiltonian $G \times H$ space

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$



o $\quad e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

basepoints b_1, b_2

$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_g(\Pi, G)$$
$$\cong G \times (U_+ \times U_-)^k \times H$$

Thm (arXiv 0203.*****)

$\tilde{\mathcal{M}}_B$ is a quasi-Hamiltonian $G \times H$ space

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

Thm (arXiv 0203.****)

$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $G = GL_2(\mathbb{C})$

$Q = A/z^k$, $A = \begin{pmatrix} a & \\ & b \end{pmatrix}$ $a \neq b$

Thm (arXiv 0203.****)

$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

$$(C, \underset{\sim}{S}, h) \qquad \underset{\sim}{S} = (S_1, \dots, S_{2k}) \qquad S_{odd/even} \in U_{+/-}$$

Moment map $\mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, h^{-1}) \in G \times H$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

Thm (arXiv 0203.****)

$$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

$$\ni (C, \underset{\sim}{S}, h) \qquad \underset{\sim}{S} = (S_1, \ldots, S_{2k}) \qquad S_{\text{odd/even}} \in U_{+/-}$$

Moment map $\mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, \; h^{-1}) \in G \times H$

Cor. $\mathcal{B}(Q) := \mathcal{A}(Q) /\!/ G$ is a quasi-Hamiltonian H-space

$$= \mu_G^{-1}(1)/G \qquad = \widetilde{\mathcal{M}}_B((\mathbb{P}^1, 0, Q))$$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

$$Q = A/z^k, \quad A = \begin{pmatrix} a & \\ & b \end{pmatrix} \quad a \neq b$$

Thm (arXiv 0203.****)

$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

$$\begin{array}{ccc} \ni & & \\ (C, & \underset{\sim}{S}, & h) \end{array} \quad \underset{\sim}{S} = (S_1, \dots, S_{2k}) \quad S_{odd/even} \in U_{+/-}$$

Moment map $\mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, \ h^{-1}) \in G \times H$

Cor. $\mathcal{B}(Q) := \mathcal{A}(Q) /\!\!/ G$ is a quasi-Hamiltonian H-space

$$= \mu_G^{-1}(1) / G \qquad = \widetilde{\mathcal{M}}_B((\mathbb{P}^1, 0, Q))$$

$$\cong \{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$$

Wild Character Varieties

Cor.

$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$ is a quasi-Hamiltonian H-space

Wild Character Varieties

Cor.

$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$ is a quasi-Hamiltonian H-space

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G \}$$

Wild Character Varieties

Cor.

$$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \} \text{ is a quasi-Hamiltonian } H\text{-space}$$

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G \}$$

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \} \quad \text{(Gauss)}$$

Wild Character Varieties

Cor.

$$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$$ is a quasi-Hamiltonian H-space

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G \}$$

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \} \qquad (\text{Gauss})$$

E.g. $k=2$ $$\left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right)_{11} = 1 + ab$$

Wild Character Varieties

Cor.

$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$ is a quasi-Hamiltonian H-space

$$\cong \{ (S_2, \dots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G \}$$

$$\cong \{ (S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \} \qquad (\text{Gauss})$$

E.g. $k=2$ $\quad \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \right)_{11} = 1 + ab$

so $\mathcal{B}(Q) \cong \mathcal{B}(V)$ of Van den Bergh

$$\mu = h^{-1} = \left(1+ab, (1+ba)^{-1} \right)$$

Wild Character Varieties

Cor.

$$\{(\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1\}$$ is a quasi-Hamiltonian H-space

$$\cong \{(S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G\}$$

$$\cong \{(S_2, \ldots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0\} \quad \text{(Gauss)}$$

E.g. $k=2$ $\quad \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right)_{11} = 1 + ab$

so $\mathcal{B}(Q) \cong \mathcal{B}(V)$ of Van den Bergh

$$\mu = h^{-1} = \left(1+ab, (1+ba)^{-1}\right)$$

Lemma

$$\left(\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_r \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b_r & 1 \end{pmatrix}\right)_{11} = (a_1, b_1, \ldots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

Wild Character Varieties

Cor.

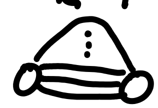
$$\{(\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1\}$$ is a quasi-Hamiltonian H-space

$$\cong \{(S_2, \dots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G\}$$

$$\cong \{(S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0\} \qquad (\text{Gauss})$$

$$\cong \{\underset{\sim}{a}, \underset{\sim}{b} \in \mathrm{Rep}(\Gamma, V) \mid (a_1, b_1, \dots, a_{k-1}, b_{k-1}) \neq 0\}$$

$k-1$

$\Gamma =$  , $V = \mathbb{C} \oplus \mathbb{C}$

Lemma

$$\left(\begin{pmatrix}1 & a_1\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_1 & 1\end{pmatrix} \cdots \begin{pmatrix}1 & a_r\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ b_r & 1\end{pmatrix}\right)_{11} = (a_1, b_1, \dots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

Wild Character Varieties

Cor.

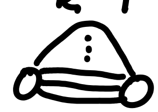
$$\{ (\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1 \}$$ is a quasi-Hamiltonian H-space

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G \}$$

$$\cong \{ (S_2, \ldots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \} \qquad \text{(Gauss)}$$

$$\cong \{ \underset{\sim}{a}, \underset{\sim}{b} \in \mathrm{Rep}(\Gamma, V) \mid (a_1, b_1, \ldots, a_{k-1}, b_{k-1}) \neq 0 \}$$

$$=: \mathrm{Rep}^*(\Gamma, V)$$

$\Gamma =$ 

, $V = \mathbb{C} \oplus \mathbb{C}$

Lemma

$$\left(\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_r & 1 \end{pmatrix} \right)_{11} = (a_1, b_1, \ldots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

Wild Character Varieties

Cor.

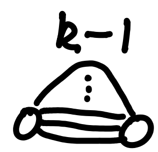
$\{(\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1\}$ is a quasi-Hamiltonian H-space

$$\cong \{(S_2, \dots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G\}$$

$$\cong \{(S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0\} \qquad \text{(Gauss)}$$

$$\cong \{\underset{\sim}{a}, \underset{\sim}{b} \in \mathrm{Rep}(\Gamma, V) \mid (a_1, b_1, \dots, a_{k-1}, b_{k-1}) \neq 0\}$$

$$=: \mathrm{Rep}^*(\Gamma, V)$$

$\Gamma =$ 

, $V = \mathbb{C} \oplus \mathbb{C}$

[Similarly for $V = V_1 \oplus V_2$ any dimension
(2009-2015)
Γ any "fission graph"]

$$\mu(a_1, \dots, b_{k-1}) = ((a_1, b_1, \dots, a_{k-1}, b_{k-1}), (b_{k-1}, \dots, b_1, a_1)^{-1})$$

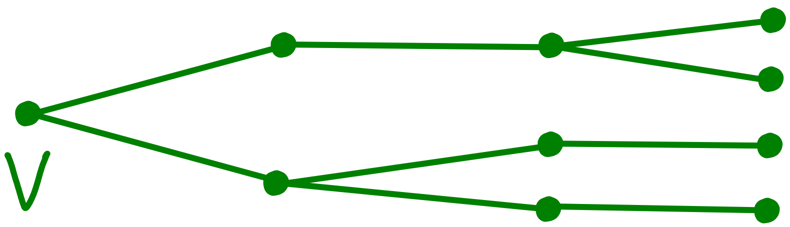
Fission graphs

$G = GL(V)$

$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in \mathfrak{t})$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3$:



"fission tree"

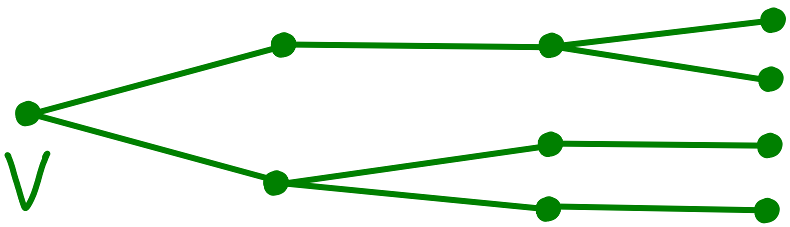
Fission graphs

$G = GL(V)$

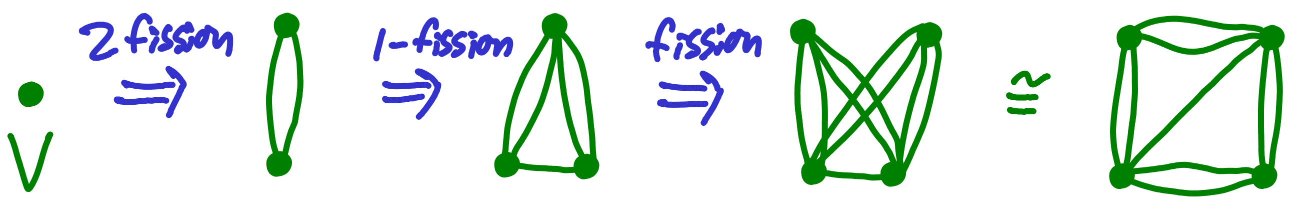
$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in \mathfrak{t})$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3$:

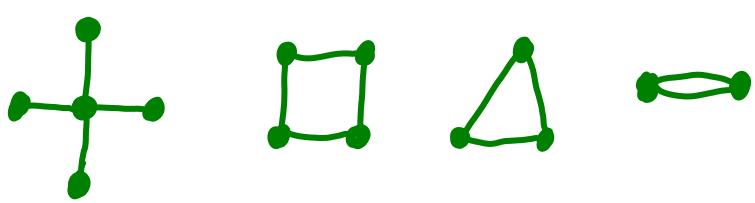


"fission tree"

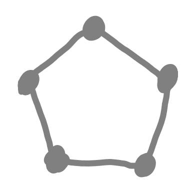


"fission graph" $\Gamma(Q)$

- $r = 2$ get all complete k-partite graphs
- e.g.



but not



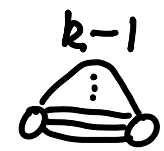
$Q = \mathrm{diag}(q_1, \ldots, q_n) \Rightarrow$ nodes $= \{1, \ldots, n\}$, # edges $i \leftrightarrow j$ $= \deg_w(q_i - q_j) - 1$

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \qquad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \widetilde{\mathcal{M}}_B \,/\!\!/_{(q_1,q_2)}\, H$$

$$= \mathrm{Rep}^*(\Gamma, V) \,/\!\!/_{(q_1,q_2)}\, H$$

$\Gamma =$ 

$, \quad V = \mathbb{C} \oplus \mathbb{C}$

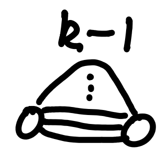
"multiplicative quiver variety"

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \widetilde{\mathcal{M}}_B \,/\!\!/_{(q_1,q_2)} H$$

$$= \mathrm{Rep}^*(\Gamma, V) \,/\!\!/_{(q_1,q_2)} H$$

$\Gamma =$ 

$, \; V = \mathbb{C} \oplus \mathbb{C}$

"multiplicative quiver variety"

E.g. $k=3$ (Painlevé 2 Betti space)

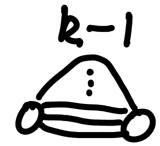
$$\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant}$$

(Flaschka-Newell surface)

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \operatorname{Rep}^*(\Gamma, V) /\!\!/_{(q_1,q_2)} H$$

$\Gamma =$ 

$, \quad V = \mathbb{C} \oplus \mathbb{C}$

"multiplicative quiver variety"

Also $\mathcal{M}^* \cong \operatorname{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ "Nakajima/additive quiver variety"

(P.B 2008, Hiroe-Yamakawa 2013)

E.g. $k=3$ (Painlevé 2 Betti space)

$$\mathcal{M}_B \cong \left\{ xyz + x + y + z = b - b^{-1} \right\} \quad b \in \mathbb{C}^* \text{ constant}$$

(Flaschka-Newell surface)

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \mathrm{Rep}^*(\Gamma, V) /\!\!/_{(q_1, q_2)} H$$

$\Gamma =$ 

, $V = \mathbb{C} \oplus \mathbb{C}$

"multiplicative quiver variety"

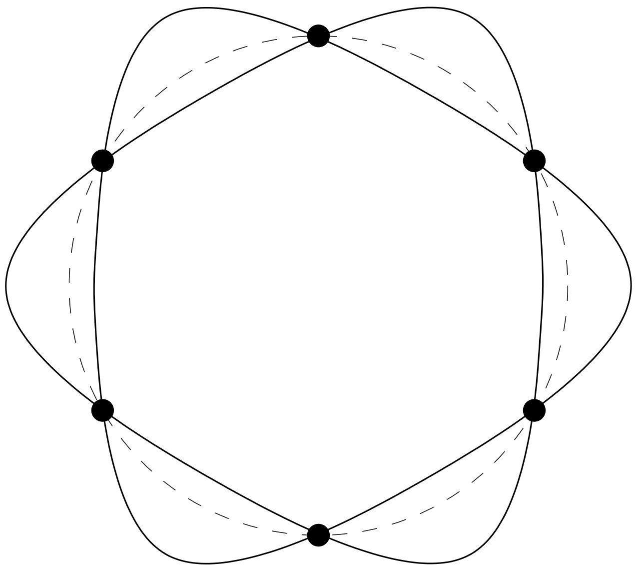
Also $\mathcal{M}^* \cong \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H$ "Nakajima/additive quiver variety"

(P.B 2008, Hiroe-Yamakawa 2013)

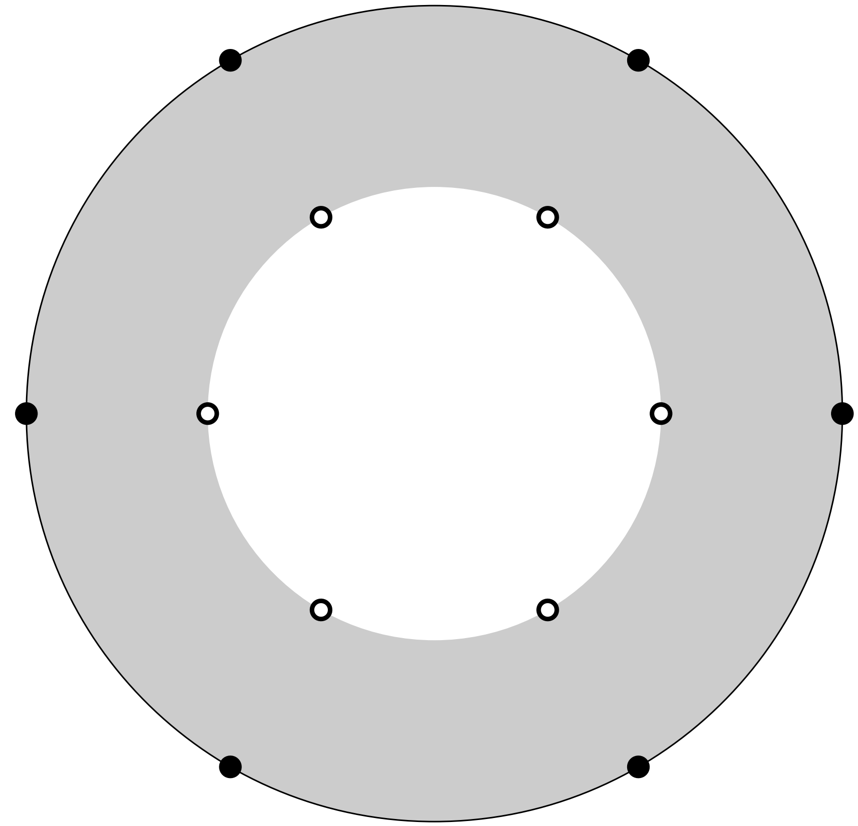
$$\boxed{\begin{array}{ccc} \mathcal{M}^* & \xRightarrow{\text{RHB}} & \mathcal{M}_B \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H & & \mathrm{Rep}^*(\Gamma, V) /\!\!/_{q} H \end{array}}$$

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



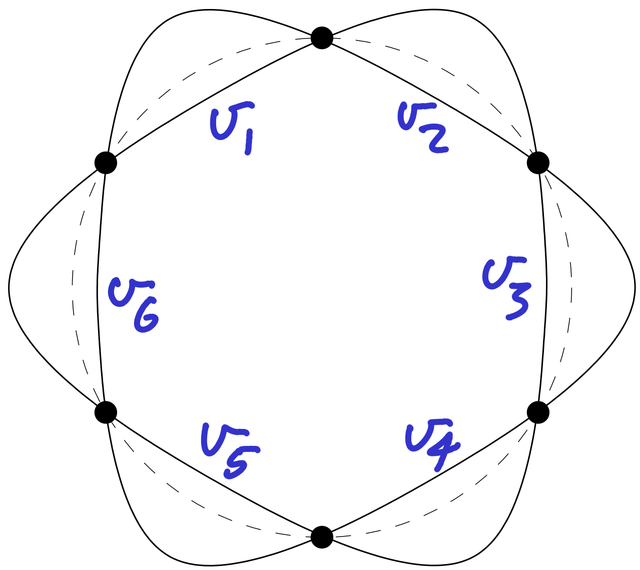
Stokes diagram with Stokes directions



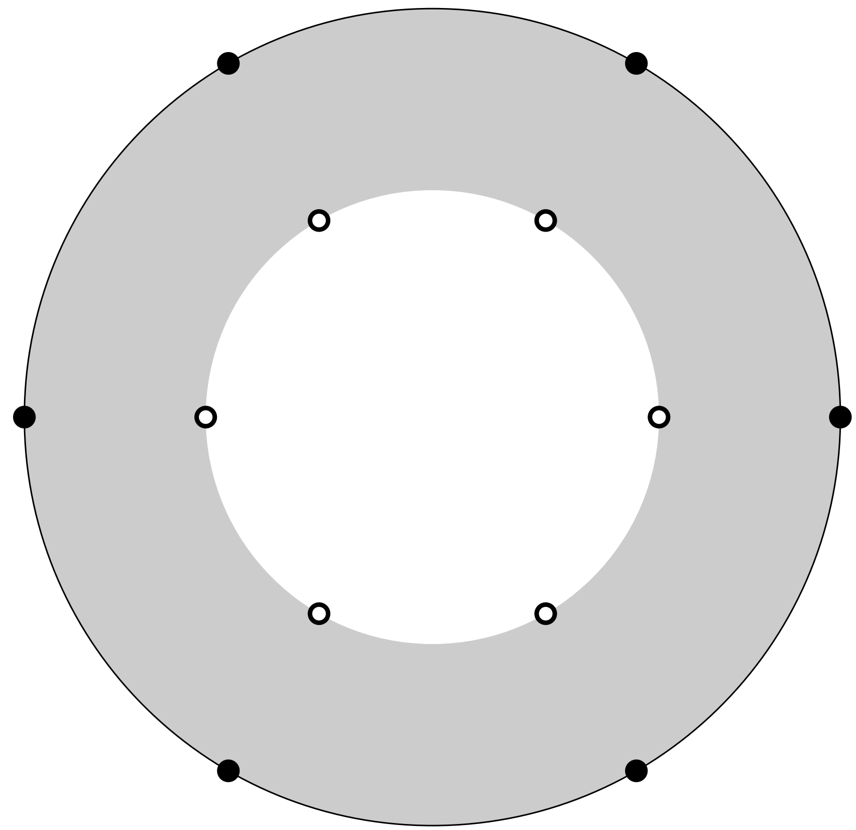
Halo at ∞ with singular directions

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions

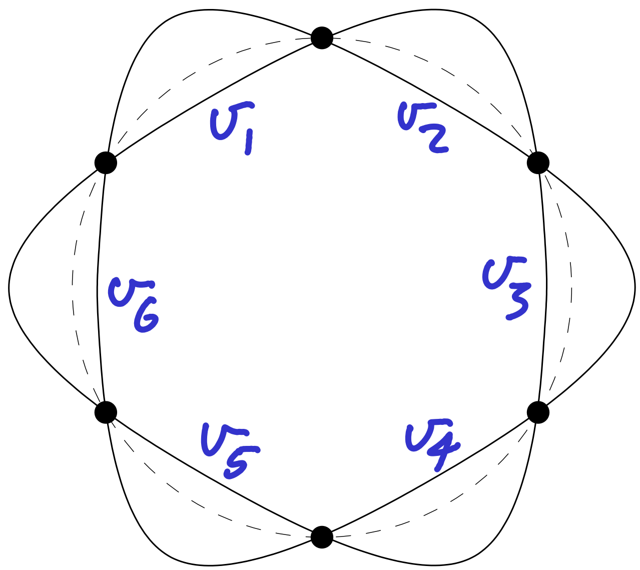


Halo at ∞ with singular directions

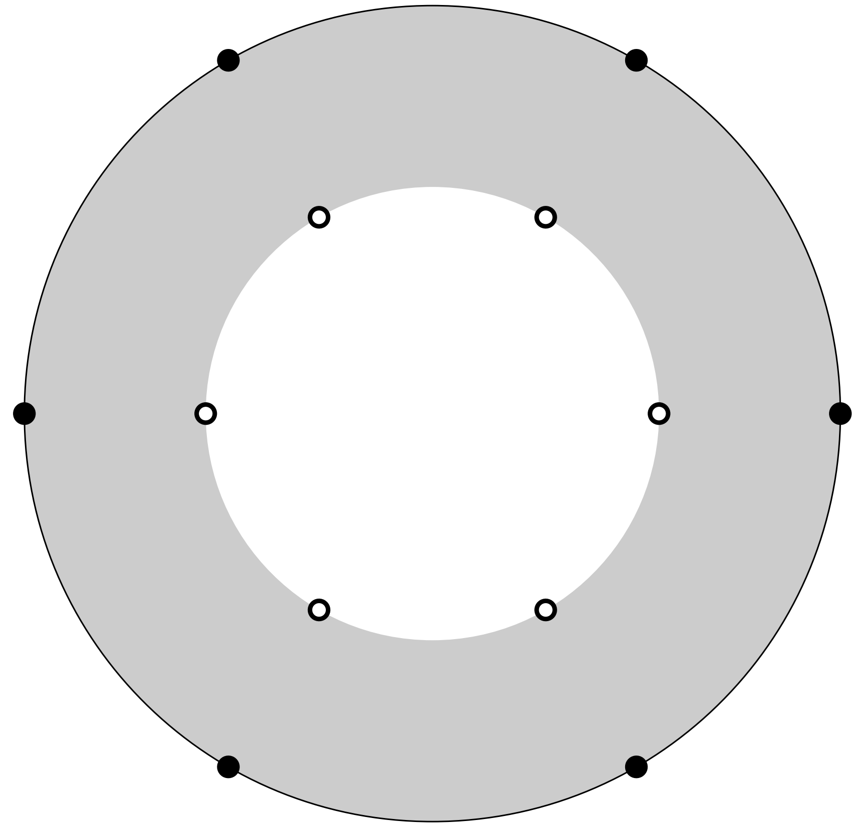
Subdominant solutions $\quad v_i \nparallel v_{i+1}$

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions



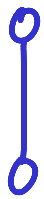
Halo at ∞ with singular directions

Subdominant solutions $\quad v_i \nparallel v_{i+1}$

$$\mathcal{M}_B \cong \{ xyz + x + y + z = b - b^{-1} \}$$

$$\cong \left\{ (p_1, \dots, p_6) \in (\mathbb{P}^1)^6 \;\middle|\; \begin{array}{c} p_i \neq p_{i+1} \bmod 6 \\ \dfrac{(p_1 - p_2)(p_3 - p_4)(p_5 - p_6)}{(p_2 - p_3)(p_4 - p_5)(p_6 - p_1)} = b^2 \end{array} \right\} \Big/ PSL_2(\mathbb{C})$$

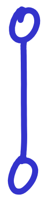
Summary



$$B_2 = B(V_1, V_2)$$

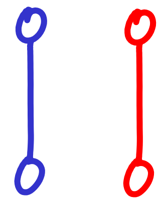
$$\mu \sim (a, b) = ab + 1$$

Summary



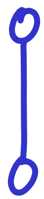
$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

$\mu \sim (a, b) = ab + 1$



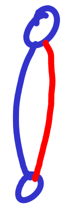
$\mathcal{B}_2 \times \mathcal{B}_2$

Summary



$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

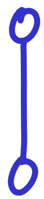
$\mu \sim (a, b) = ab + 1$



$\mathcal{B}_2 \underset{H}{\otimes} \mathcal{B}_2$

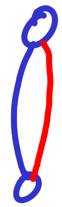
$\mu \sim (a,b)(c,d)$

Summary



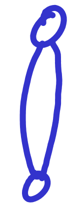
$\mathcal{B}_2 = \mathcal{B}(v_1, v_2)$

$\mu \sim (a, b) = ab + 1$



$\mathcal{B}_2 \underset{H}{\otimes} \mathcal{B}_2$

$\mu \sim (a, b)(c, d)$



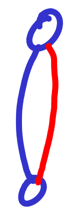
$\mathcal{B}_4$

$\mu \sim (a, b, c, d)$

Summary

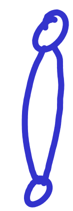
$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

$\mu \sim (a,b) = ab+1$



$\mathcal{B}_2 \otimes_H \mathcal{B}_2$

$\mu \sim (a,b)(c,d)$



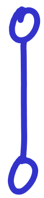
$\mathcal{B}_4$

$\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$

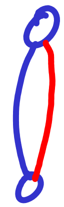
$c' = (a,b)^{-1}(a,b,c)$

Summary



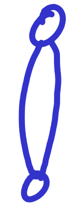
$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

$\mu \sim (a,b) = ab+1$



$\mathcal{B}_2 \otimes_H \mathcal{B}_2$

$\mu \sim (a,b)(c,d)$



$\mathcal{B}_4$

$\mu \sim (a,b,c,d)$

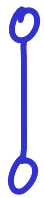
Continuants factorise:

$$\begin{aligned}(a,b,c,d) &= (a,b)(c',d) \\ &= (a,b')(c,d)\end{aligned}$$

$c' = (a,b)^{-1}(a,b,c)$

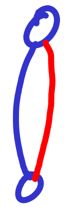
$b' = (b,c,d)(c,d)^{-1}$

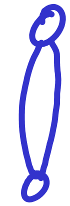
Summary



$$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$$

$$\mu \sim (a, b) = ab + 1$$





$$\mathcal{B}_2 \otimes_H \mathcal{B}_2 \; \underset{R}{\overset{L}{\rightrightarrows}} \; \mathcal{B}_4$$

$$\mu \sim (a,b)(c,d) \qquad\qquad \mu \sim (a,b,c,d)$$

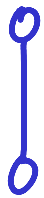
Continuants factorise: $(a, b, c, d) = (a,b)(c', d)$

$$= (a, b')(c, d)$$

$$c' = (a,b)^{-1}(a,b,c)$$

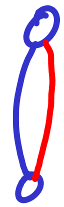
$$b' = (b,c,d)(c,d)^{-1}$$

Summary



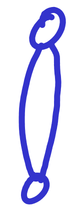
$$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$$

$$\mu \sim (a,b) = ab+1$$



$$\mathcal{B}_2 \underset{H}{\circledast} \mathcal{B}_2 \quad \underset{R}{\overset{L}{\rightrightarrows}} \quad \mathcal{B}_4$$

$$\mu \sim (a,b)(c,d)$$



$$\mu \sim (a,b,c,d)$$

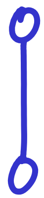
Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad c' = (a,b)^{-1}(a,b,c)$

$\qquad\qquad\qquad\qquad\qquad\qquad = (a,b')(c,d)$ $\qquad b' = (b,c,d)(c,d)^{-1}$

Thm (B.-Paluba-Yamakawa)

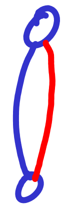
All such factorisation maps relate the quasi-Hamiltonian structures

Summary



$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

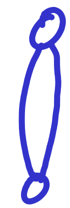
$\mu \sim (a,b) = ab+1$



$\mathcal{B}_2 \circledast_H \mathcal{B}_2$

$\mu \sim (a,b)(c,d)$

$\underset{R}{\overset{L}{\rightrightarrows}}$



$\mathcal{B}_4$

$\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$

$= (a,b')(c,d)$

$c' = (a,b)^{-1}(a,b,c)$

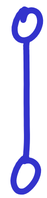
$b' = (b,c,d)(c,d)^{-1}$

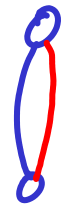
Thm (B.-Paluba-Yamakawa)

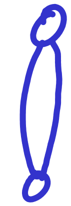
All such factorisation maps relate the quasi-Hamiltonian structures

- Count all factorisations (into linear factors) $\rightsquigarrow 14$

Summary







$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$ $\qquad$ $\mathcal{B}_2 \underset{H}{\circledast} \mathcal{B}_2 \underset{R}{\overset{L}{\rightrightarrows}} \mathcal{B}_4$

$\mu \sim (a,b) = ab+1$ $\qquad$ $\mu \sim (a,b)(c,d)$ $\qquad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad$ $c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad$ $b' = (b,c,d)(c,d)^{-1}$

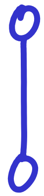
Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

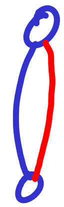
& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

Summary

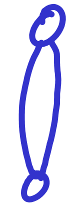


$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

$\mu \sim (a,b) = ab+1$



$\mathcal{B}_2 \circledast_H \mathcal{B}_2 \underset{R}{\overset{L}{\rightrightarrows}} \mathcal{B}_4$



$\mu \sim (a,b)(c,d)$ $\qquad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad b' = (b,c,d)(c,d)^{-1}$

Thm (B.-Paluba-Yamakawa)

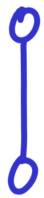
All such factorisation maps relate the quasi-Hamiltonian structures

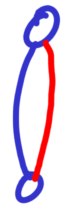
- Count all factorisations (into linear factors) $\rightsquigarrow 14$

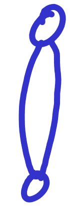
& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

- follows since L & R form "free duplicial algebra" (Loday)

Summary







$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$ $\qquad$ $\mathcal{B}_2 \circledast_H \mathcal{B}_2 \underset{R}{\overset{L}{\rightrightarrows}} \mathcal{B}_4$

$\mu \sim (a,b) = ab+1$ $\qquad$ $\mu \sim (a,b)(c,d)$ $\qquad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad$ $c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad$ $b' = (b,c,d)(c,d)^{-1}$

Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

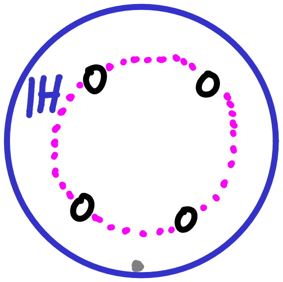
- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

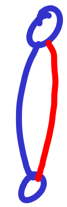
  & similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

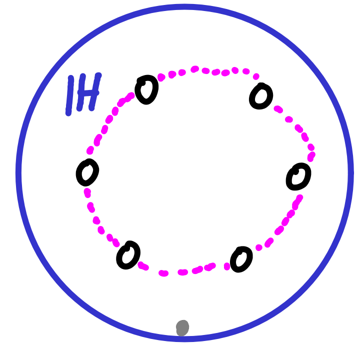
- follows since L & R form "free duplicial algebra" (Loday)

- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary







$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$ $\qquad$ $\mathcal{B}_2 \underset{H}{\circledast} \mathcal{B}_2 \; \overset{L}{\underset{R}{\rightrightarrows}} \; \mathcal{B}_4$

$\mu \sim (a,b) = ab+1$ $\qquad$ $\mu \sim (a,b)(c,d)$ $\qquad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad$ $c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad$ $b' = (b,c,d)(c,d)^{-1}$

Thm (B.–Paluba–Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

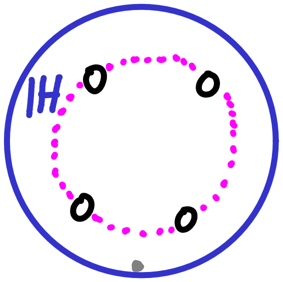
- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

- follows since L & R form "free duplicial algebra" (Loday)

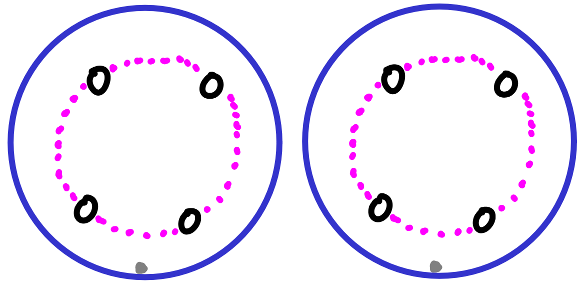
- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary



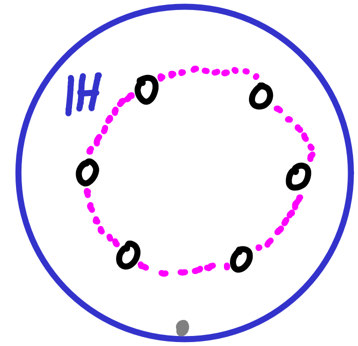
$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$

$\mu \sim (a,b) = ab+1$



$\mathcal{B}_2 \times \mathcal{B}_2 \underset{R}{\overset{L}{\rightrightarrows}} \mathcal{B}_4$

$\mu \sim (a,b)(c,d)$



$\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\quad c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\quad b' = (b,c,d)(c,d)^{-1}$

Thm (B.–Paluba–Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

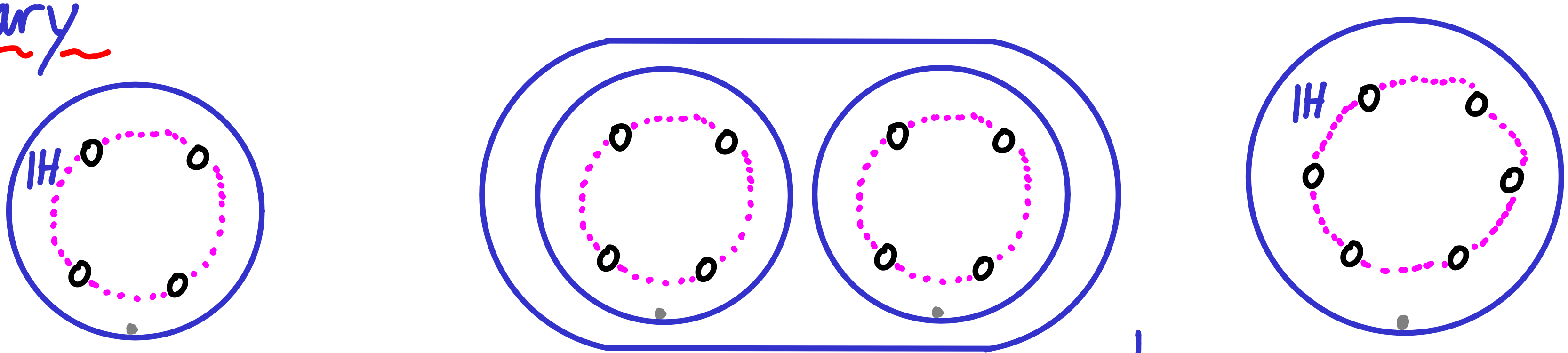
- Count all factorisations (into linear factors) $\rightsquigarrow 14$

& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

- follows since L & R form "free duplicial algebra" (Loday)

- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary



$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$ $\quad$ $\mathcal{B}_2 \circledast_H \mathcal{B}_2 \underset{R}{\overset{L}{\rightrightarrows}} \mathcal{B}_4$

$\mu \sim (a,b) = ab+1$ $\quad$ $\mu \sim (a,b)(c,d)$ $\quad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\quad$ $c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\quad$ $b' = (b,c,d)(c,d)^{-1}$

Thm (B.–Paluba–Yamakawa)

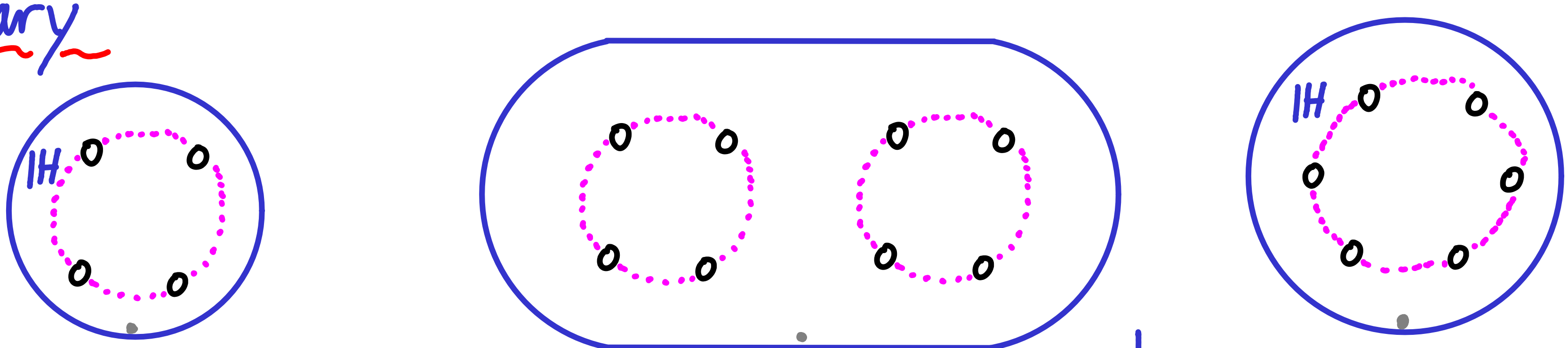
All such factorisation maps relate the quasi-Hamiltonian structures

- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

- follows since L & R form "free duplicial algebra" (Loday)
- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary



$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$ $\qquad$ $\mathcal{B}_2 \circledast_H \mathcal{B}_2 \;\substack{\xrightarrow{L} \\ \xrightarrow[R]{}}\; \mathcal{B}_4$

$\mu \sim (a,b) = ab+1$ $\qquad$ $\mu \sim (a,b)(c,d)$ $\qquad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad$ $c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad$ $b' = (b,c,d)(c,d)^{-1}$

Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

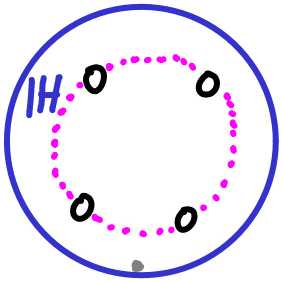
- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

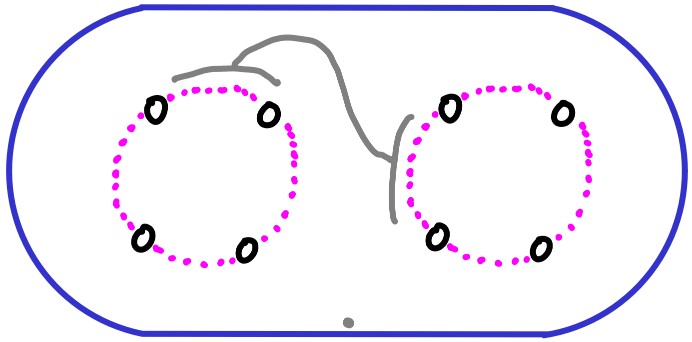
& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

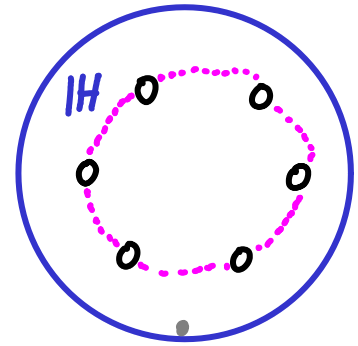
- follows since L & R form "free duplicial algebra" (Loday)

- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary







$$\mathcal{B}_2 = \mathcal{B}(V_1, V_2) \qquad \mathcal{B}_2 \circledast_H \mathcal{B}_2 \quad \overset{L}{\underset{R}{\rightrightarrows}} \quad \mathcal{B}_4$$

$$\mu \sim (a,b) = ab+1 \qquad \mu \sim (a,b)(c,d) \qquad \mu \sim (a,b,c,d)$$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad b' = (b,c,d)(c,d)^{-1}$

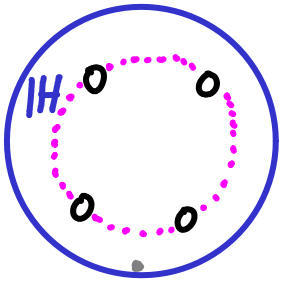
Thm (B.–Paluba–Yamakawa)

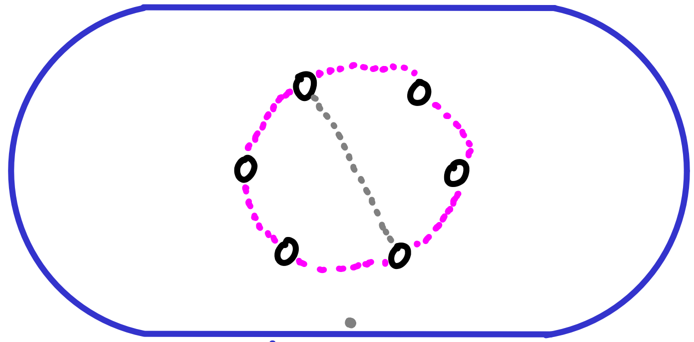
All such factorisation maps relate the quasi-Hamiltonian structures

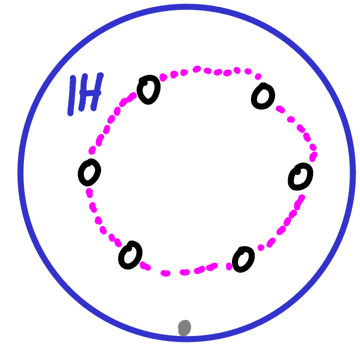
- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

  & similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)
- follows since L & R form "free duplicial algebra" (Loday)
- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

Summary







$\mathcal{B}_2 = \mathcal{B}(V_1, V_2)$ $\qquad$ $L(\mathcal{B}_2 \circledast_H \mathcal{B}_2)$ $\quad \subset \quad$ $\mathcal{B}_4$

$\mu \sim (a,b) = ab+1$ $\qquad$ $\mu \sim (a,b)(c,d)$ $\qquad$ $\mu \sim (a,b,c,d)$

Continuants factorise: $(a,b,c,d) = (a,b)(c',d)$ $\qquad c' = (a,b)^{-1}(a,b,c)$

$= (a,b')(c,d)$ $\qquad b' = (b,c,d)(c,d)^{-1}$

Thm (B.-Paluba-Yamakawa)

All such factorisation maps relate the quasi-Hamiltonian structures

- Count all factorisations (into linear factors) $\rightsquigarrow$ 14

& similarly $\mathcal{B}_n$ has $C_n = \frac{1}{n+1}\binom{2n}{n}$ factorisations (Catalan no.)

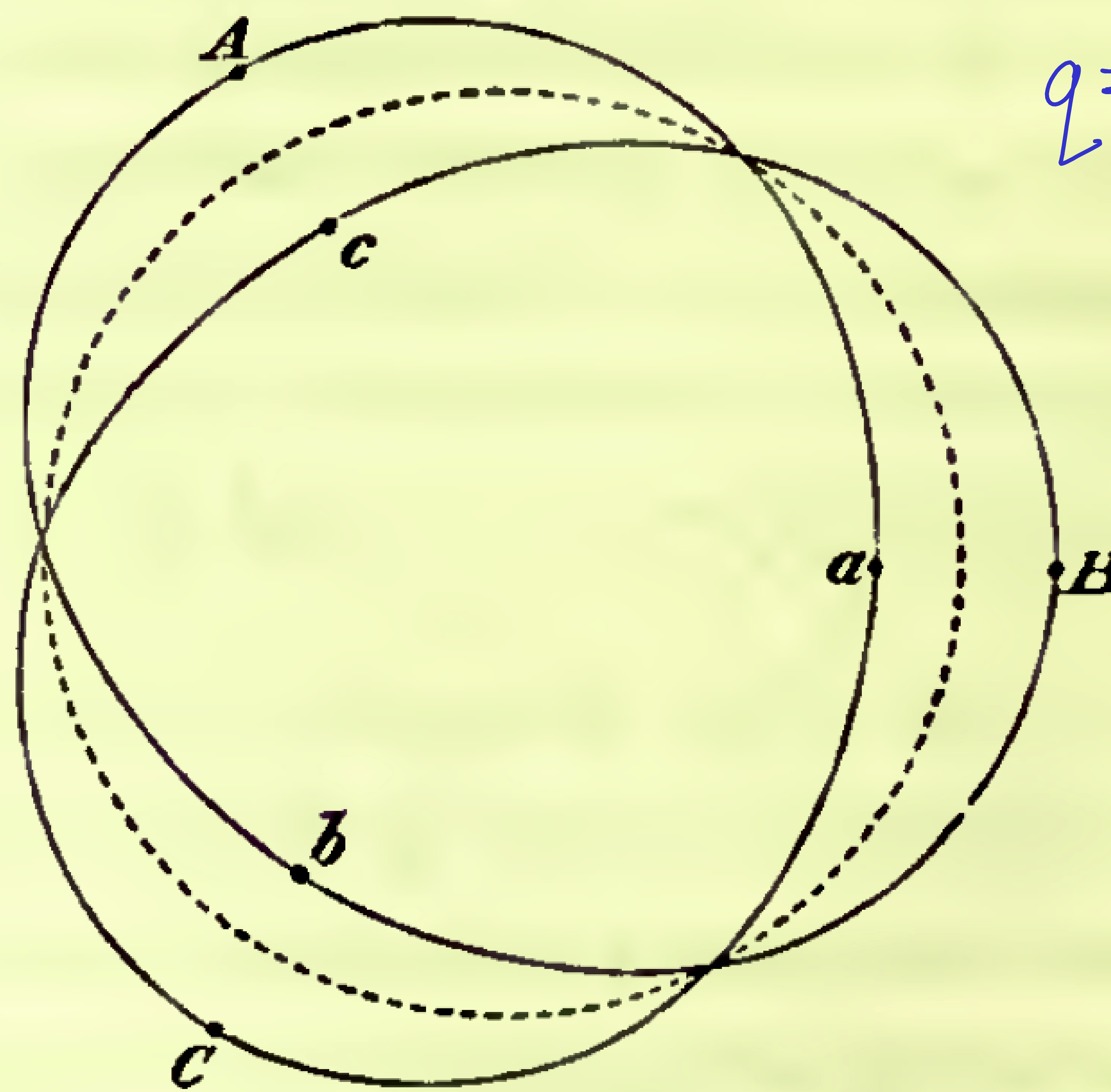
- follows since L & R form "free duplicial algebra" (Loday)

- C_n also counts triangulations of $(n+2)$-gon (Euler 1751, Segner 1758)

other the *inferior*
the existence and
different values of
ng a radius vector
e angle θ take two
and inwards from
al to the real part
perior and inferior
or in other words
nvenience suppose
l with the radius.

(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation

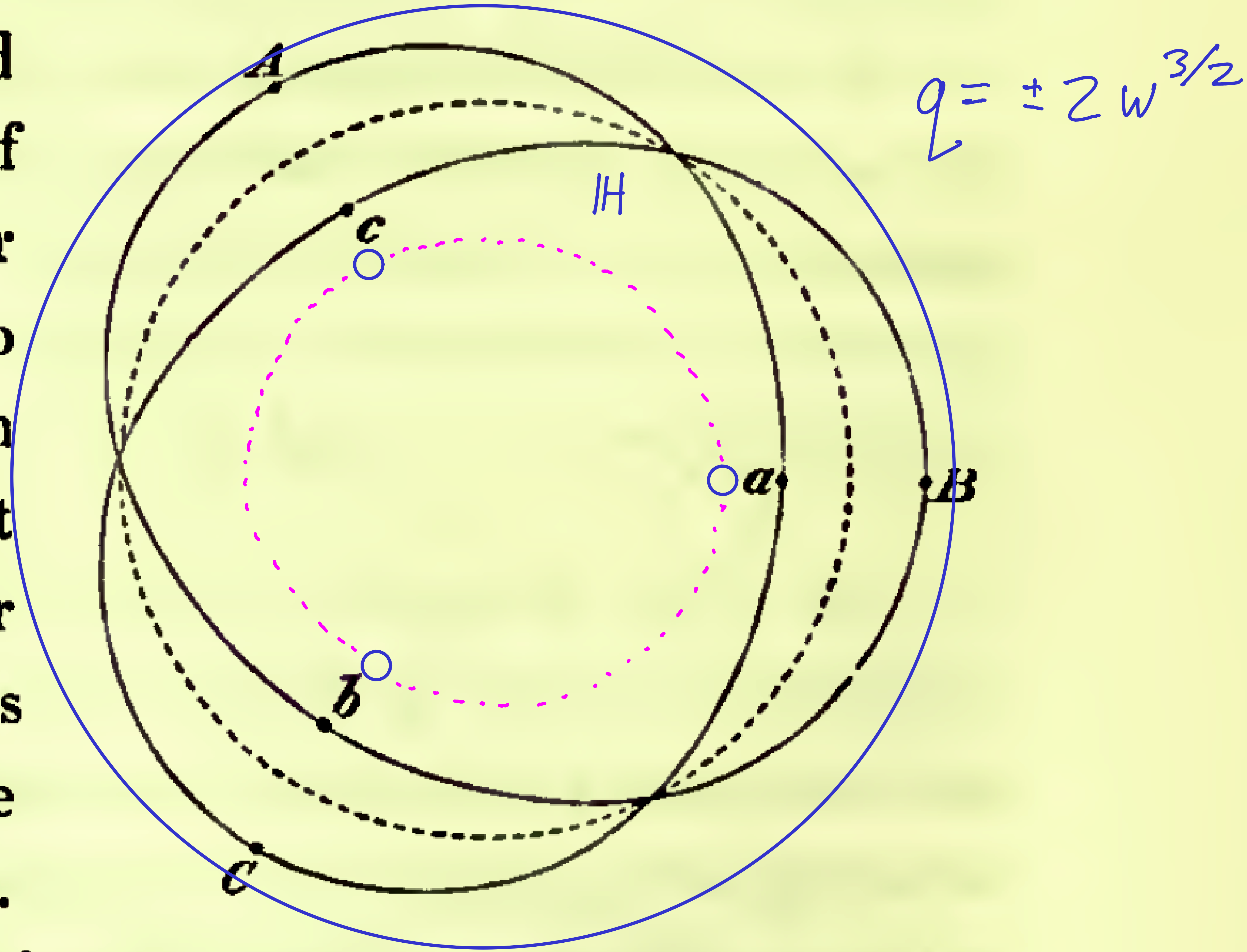
$q = \pm 2 w^{3/2}$



The curve will evidently have the form represented

other the *inferior*
the existence and
different values of
ng a radius vector
e angle θ take two
and inwards from
al to the real part
perior and inferior
or in other words
nvenience suppose
d with the radius.

(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation



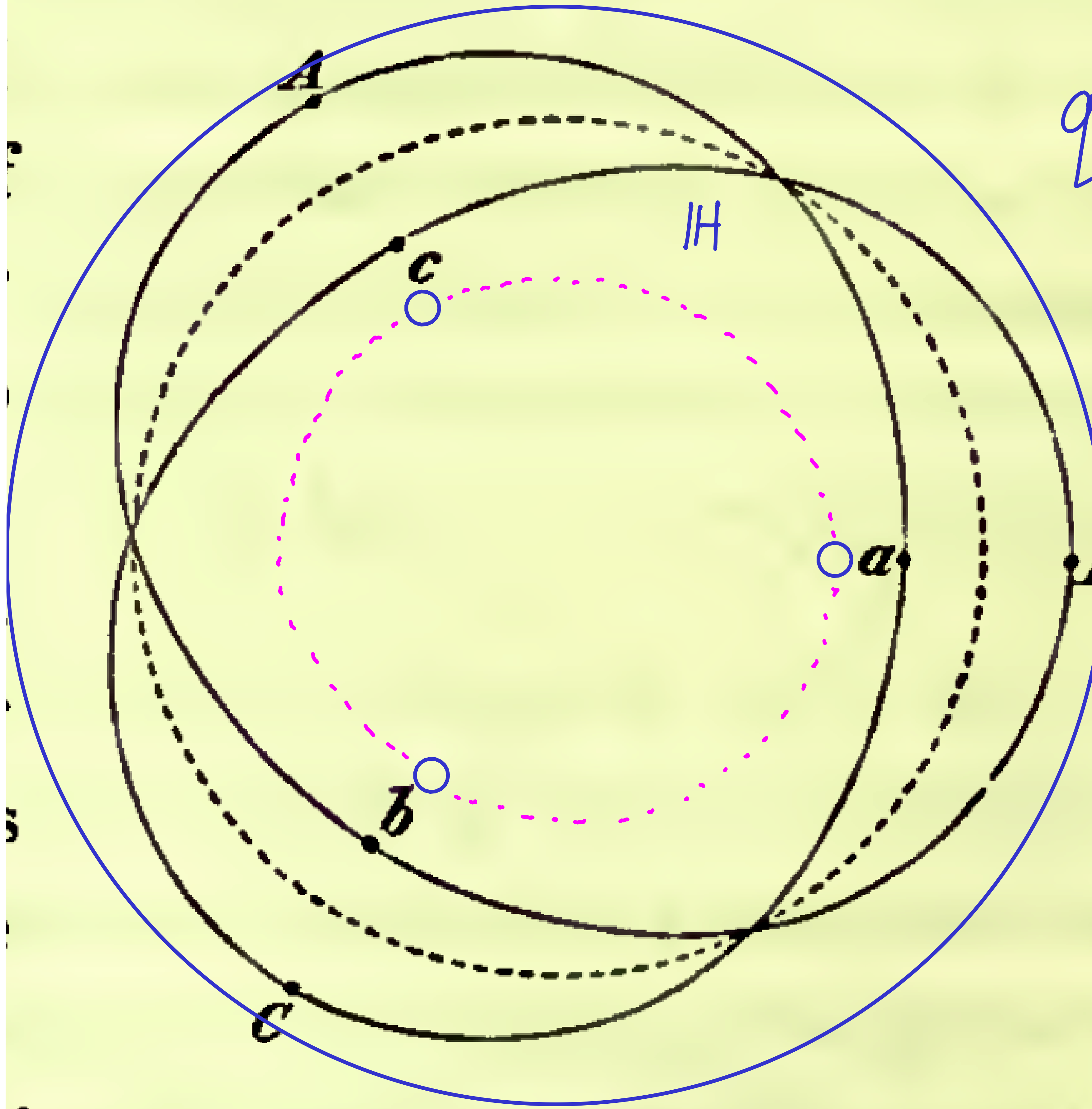
The curve will evidently have the form represented

Thm (B.-Yamakawa, arxiv:1512)

- Can define twisted stokes local systems (any reductive G)

(Stokes structures already known GL_n)

(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation



idently have the form represented

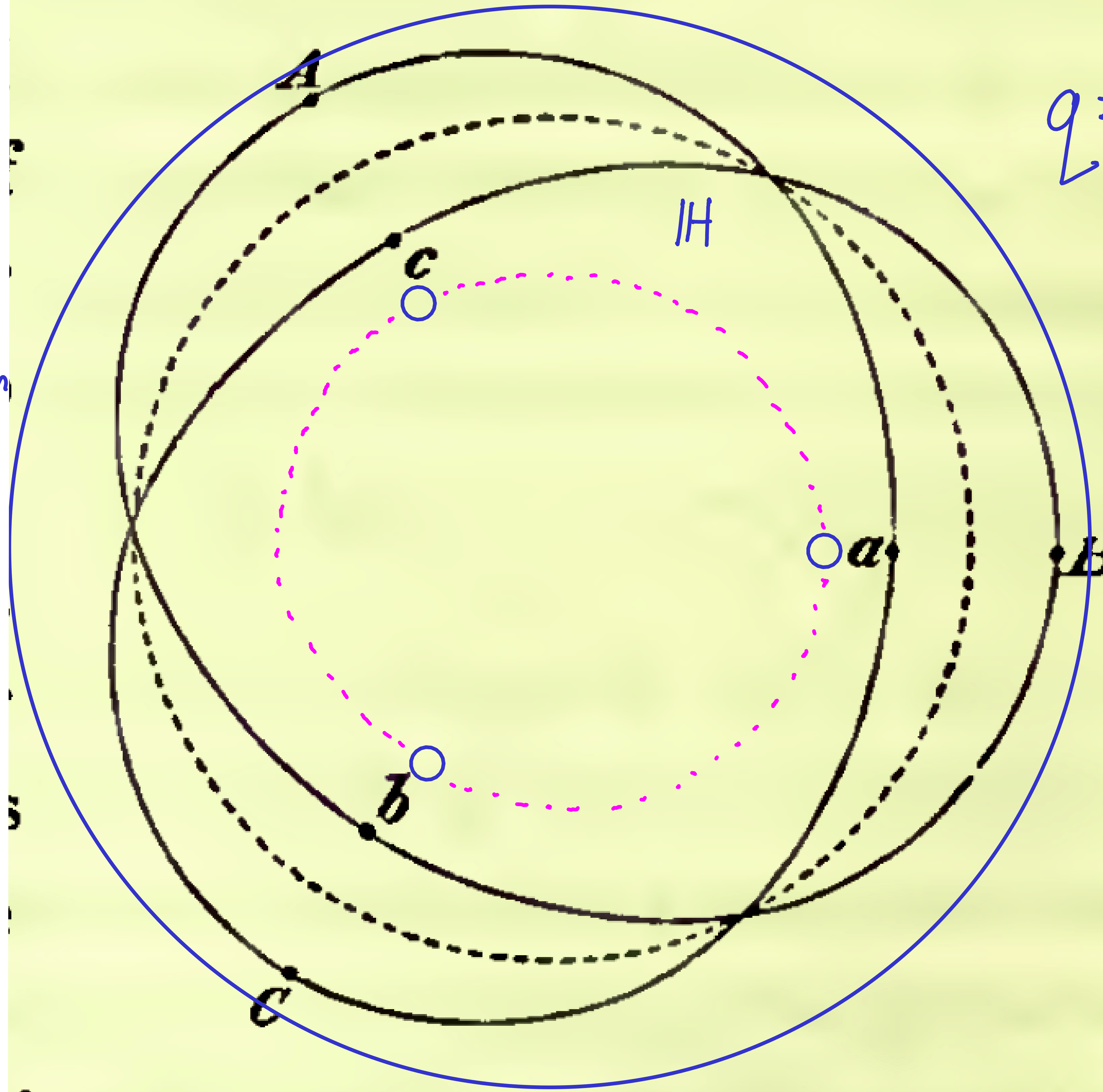
(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation

Thm (B.-Yamakawa, arxiv:1512)

- Can define twisted Stokes local systems (any reductive G)

(Stokes structures already known GL_n)

- Moduli spaces of framed twisted Stokes local systems are (twisted) quasi-Hamiltonian



$q = \pm 2 w^{3/2}$

idently have the form represented

Thm (B.-Yamakawa, arxiv: 1512)

- Can define twisted Stokes local systems (any reductive G)

(Stokes structures already known GL_n)

- Moduli spaces of framed twisted Stokes local systems are (twisted) quasi-Hamiltonian

– completes project of understanding "symplectic nature of wild π_1"

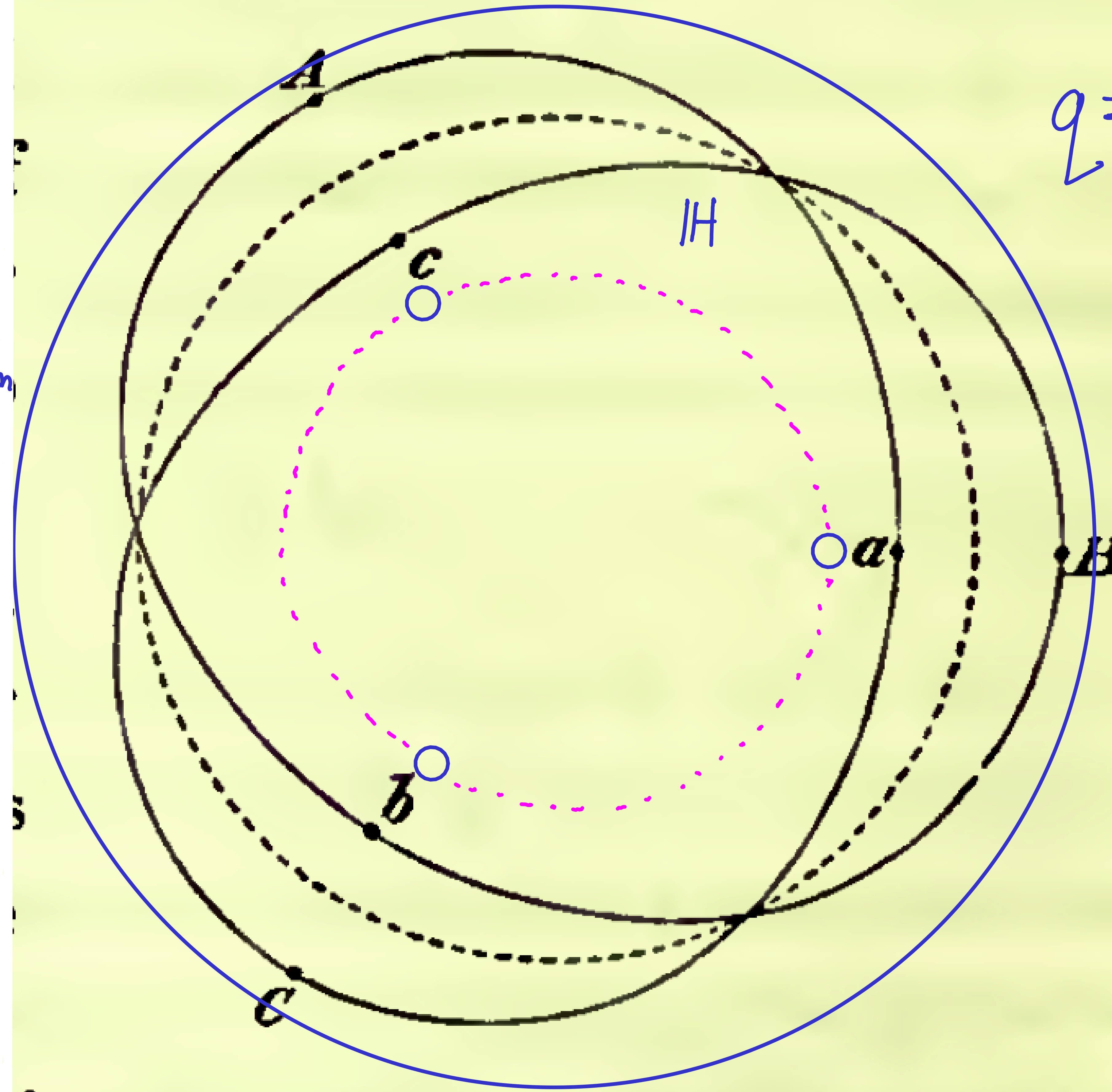
$\rightsquigarrow$ $\mathcal{B}_1 \cong GL(V_1)$ $\quad \mu \sim (a)$

$\mathcal{B}_3 \cong \{a,b,c \in \mathrm{End}(V_1) \mid \det(a,b,c) \neq 0\}$

$\mu \sim (a.b,c)$

$\vdots$

(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation



idently have the form represented

Thm (B.-Yamakawa, arxiv:1512)

- Can define twisted Stokes local systems (any reductive G)

(Stokes structures already known GL_n)

- Moduli spaces of framed twisted Stokes local systems are (twisted) quasi-Hamiltonian

– completes project of understanding "symplectic nature of wild π_1"

$\rightsquigarrow$ $\mathcal{B}_1 \cong GL(V_1)$ $\quad \mu \sim (a)$

$\mathcal{B}_3 \cong \{a,b,c \in \mathrm{End}(V_1) \mid \det(a,b,c) \neq 0\}$

$\mu \sim (a,b,c)$

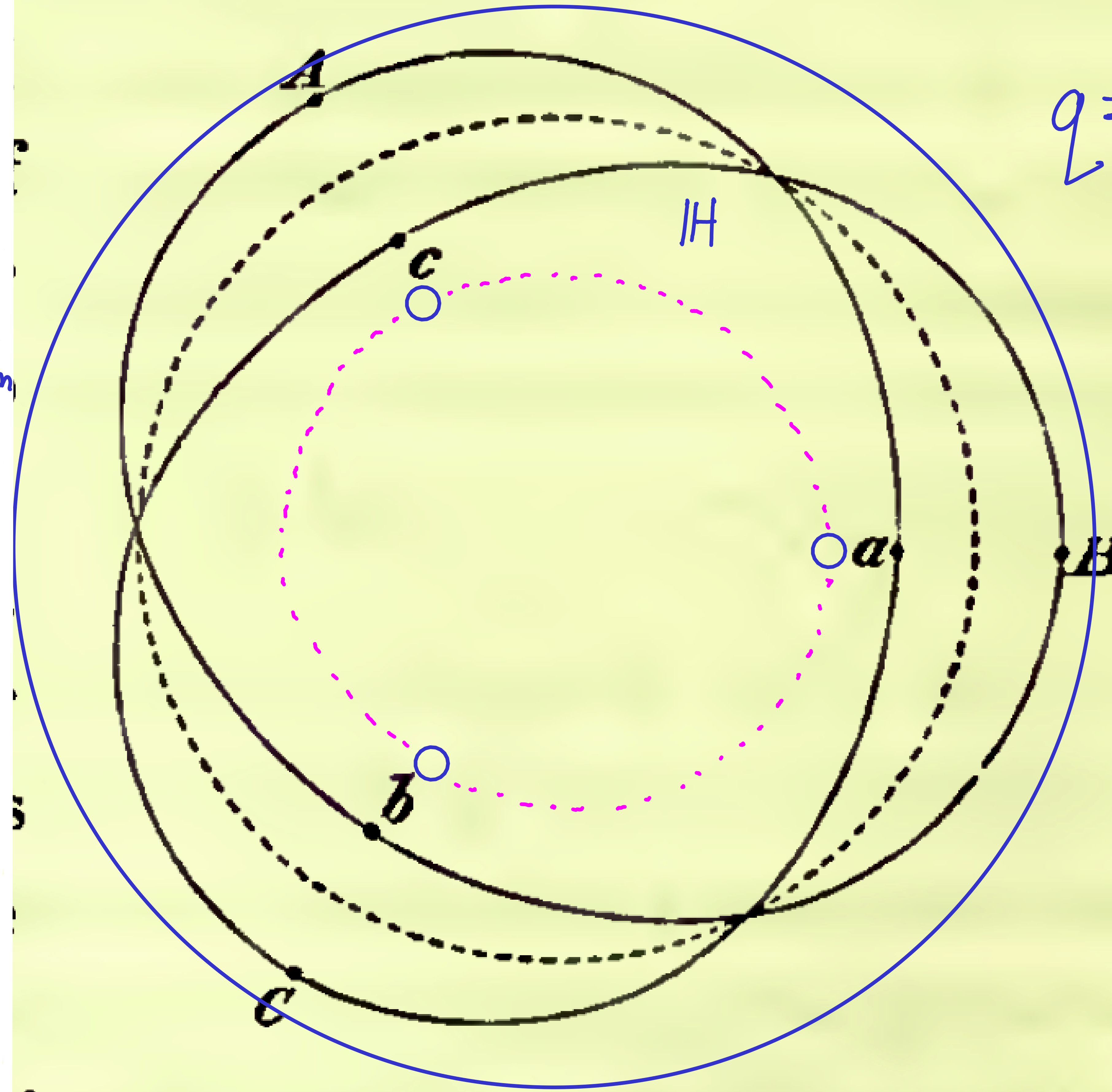
$\vdots$

Can now glue these Airy triangles ($\mathcal{B}_1$) as before, so clearly

factorisations $\Longleftrightarrow$ triangulations

$\mathcal{B}_1^n \hookrightarrow \mathcal{B}_n$

(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation



$q = \pm 2 w^{3/2}$

idently have the form represented

Thm (B.-Yamakawa, arxiv:1512)

- Can define twisted stokes local systems (any reductive G)

(Stokes structures already known GL_n)

- Moduli spaces of framed twisted stokes local systems are (twisted) quasi-Hamiltonian

– completes project of understanding "symplectic nature of wild π_1"

$\rightsquigarrow$ $\mathcal{B}_1 \cong GL(V_1)$ $\quad \mu \sim (a)$

$\mathcal{B}_3 \cong \{a,b,c \in \mathrm{End}(V_1) \mid \det(a,b,c) \neq 0\}$

$\mu \sim (a,b,c)$

$\vdots$

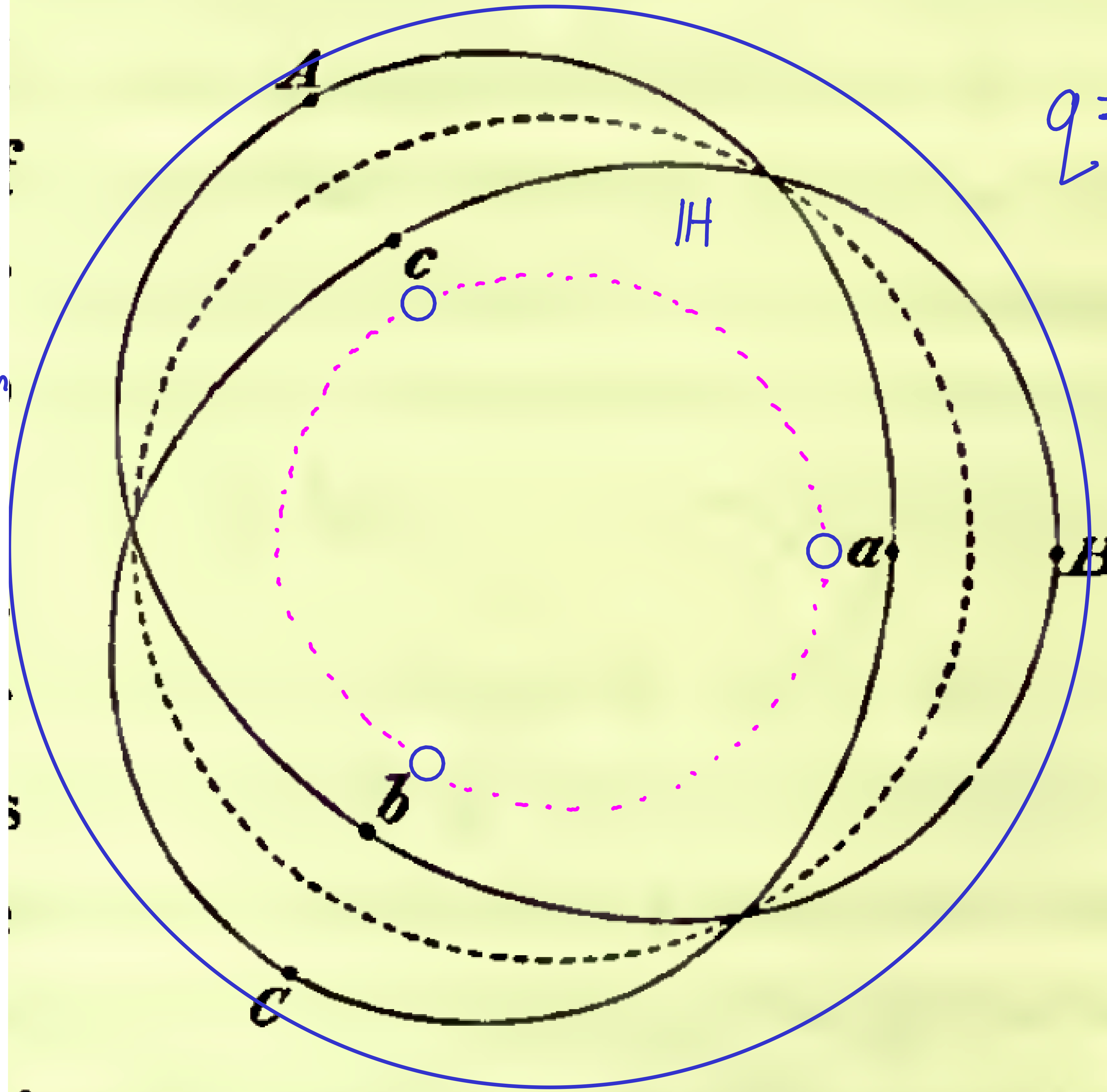
Can now glue these Airy triangles ($\mathcal{B}_1$) as before, so clearly

factorisations $\Leftrightarrow$ triangulations

$$\mathcal{B}_1^n \hookrightarrow \mathcal{B}_n$$

If $\dim(V_1) = 1$ this is familiar from complex WKB, but now see how to glue the triangles via QH fusion

(Stokes 1857) **Fig. 1.** Stokes diagram of Airy equation



...idently have the form represented

Voros '83

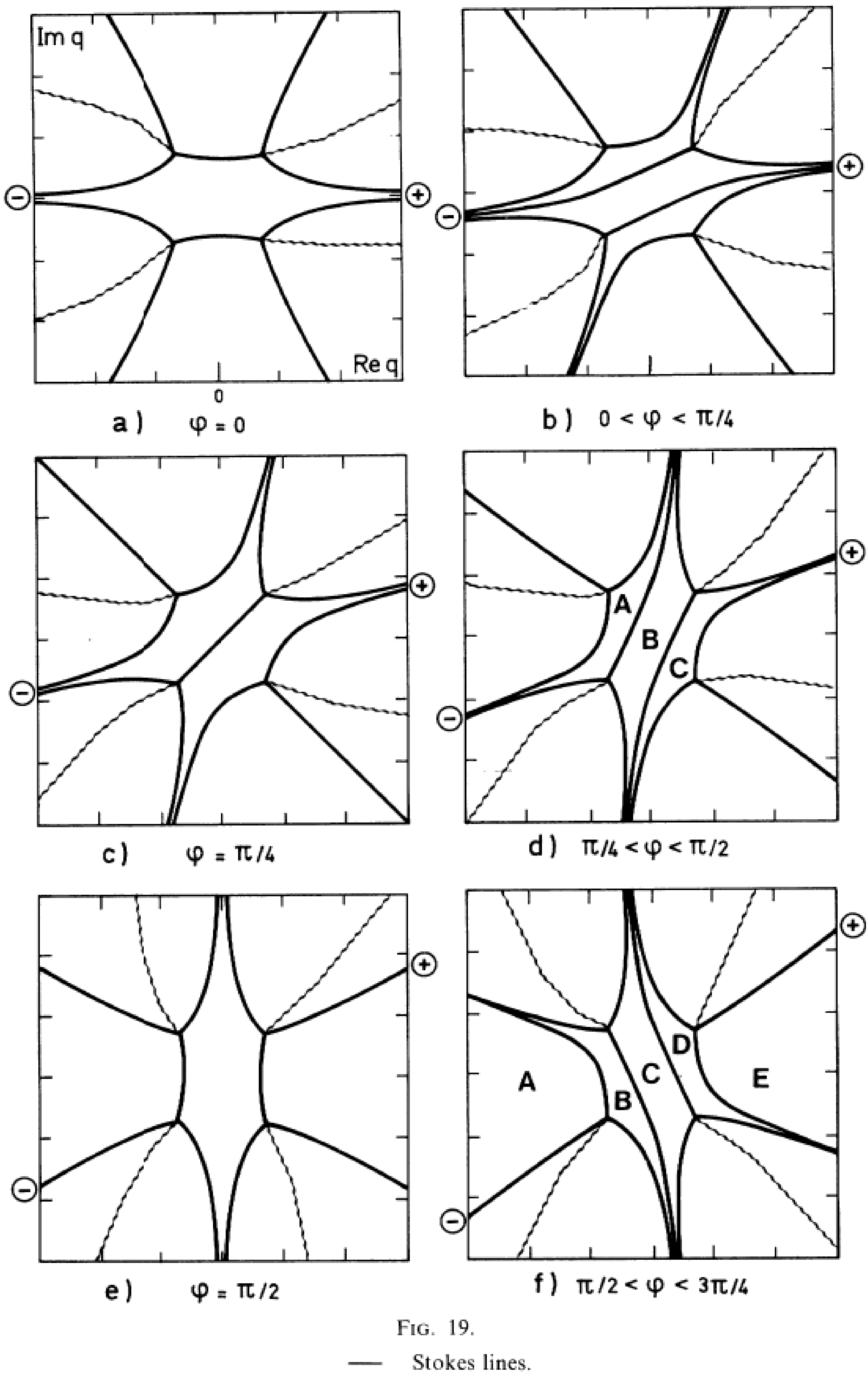
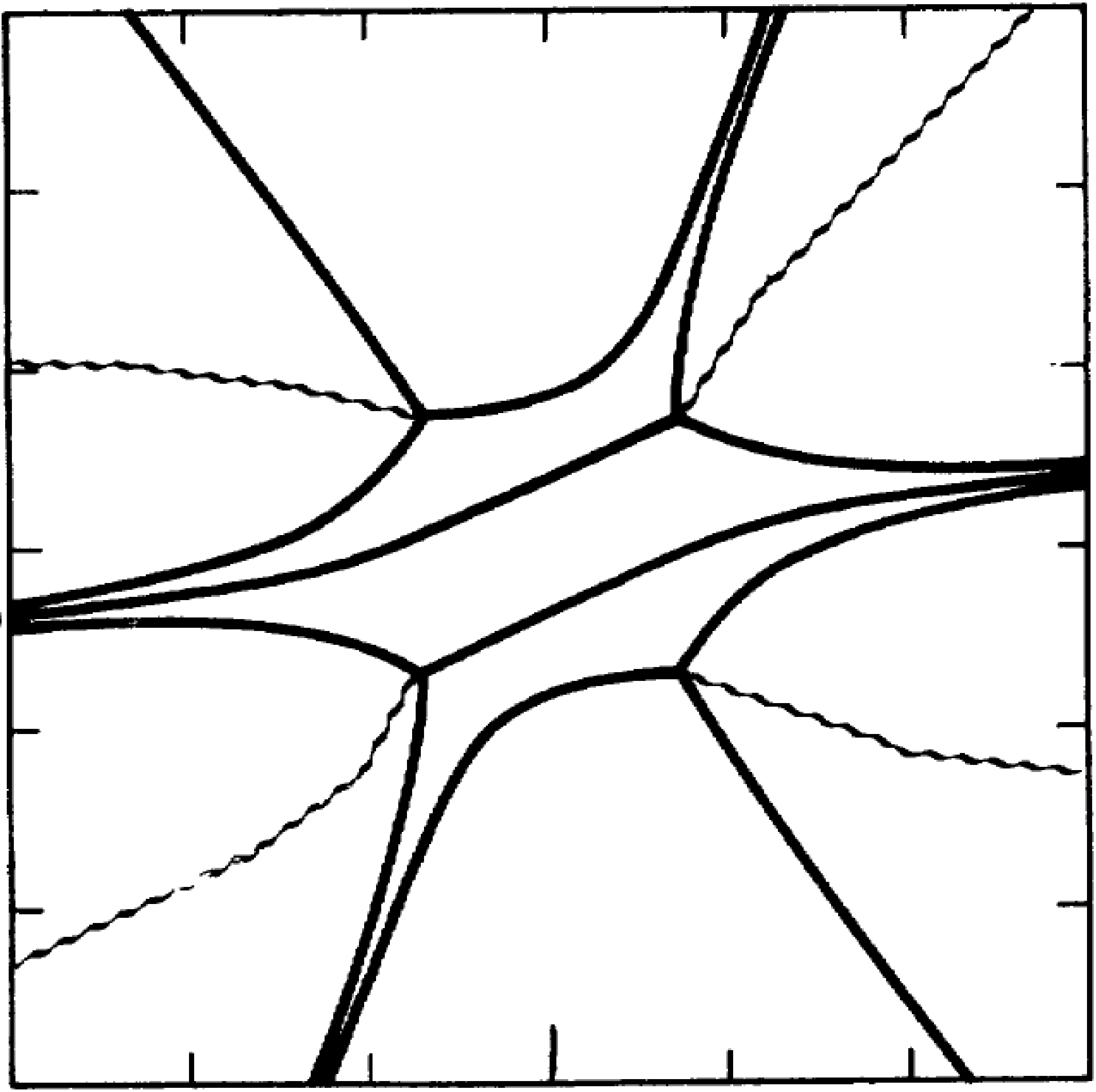
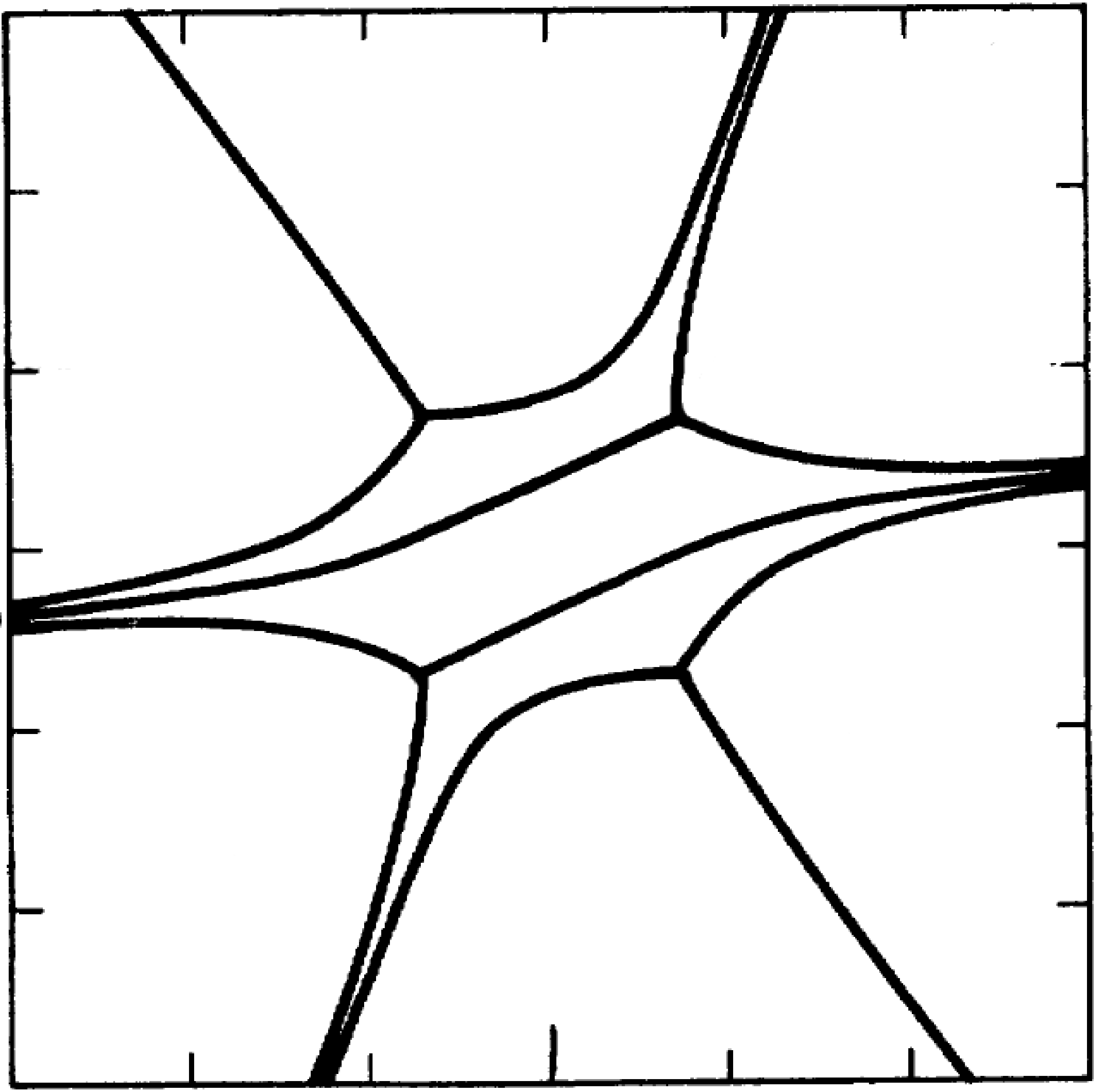
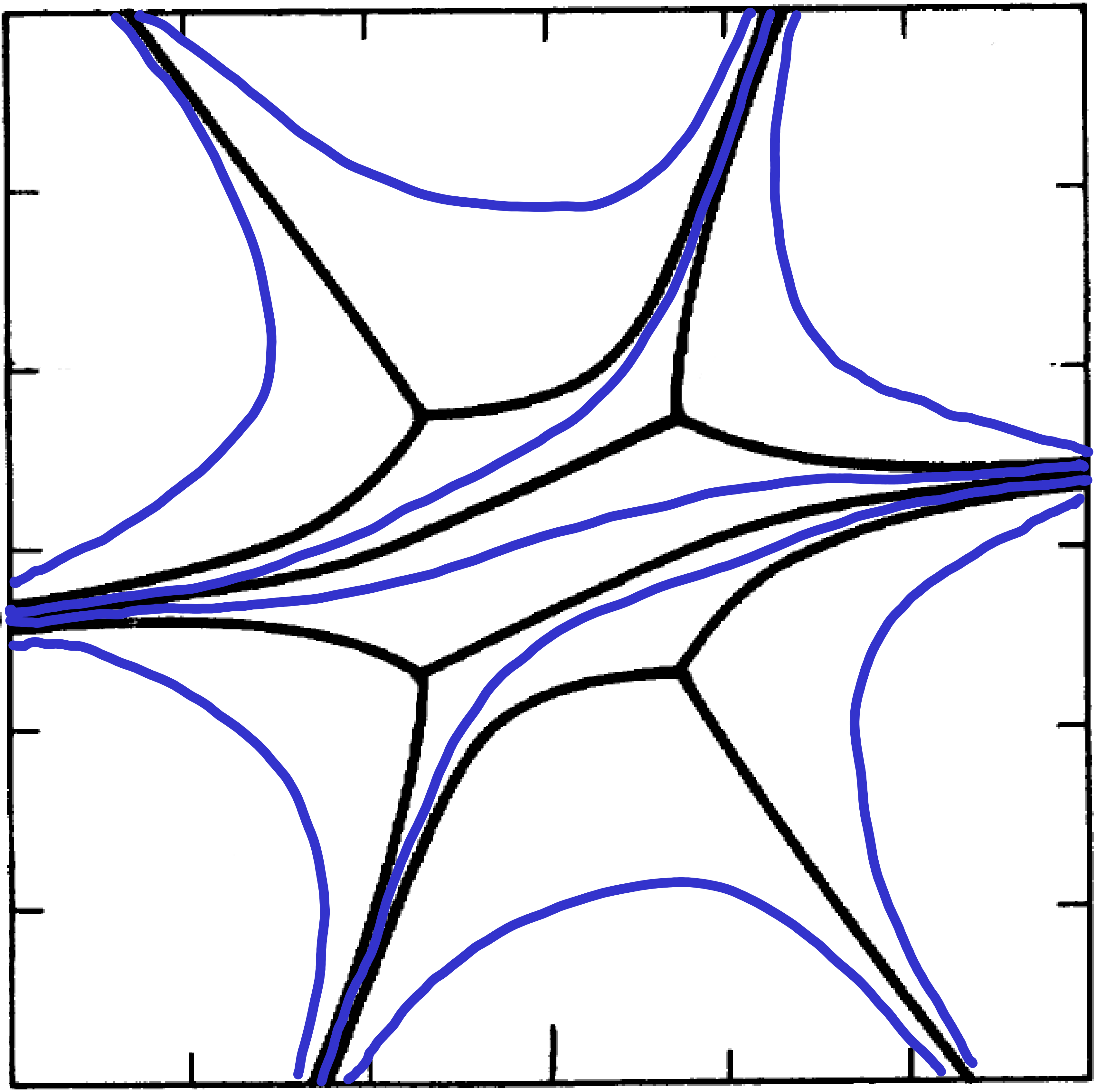


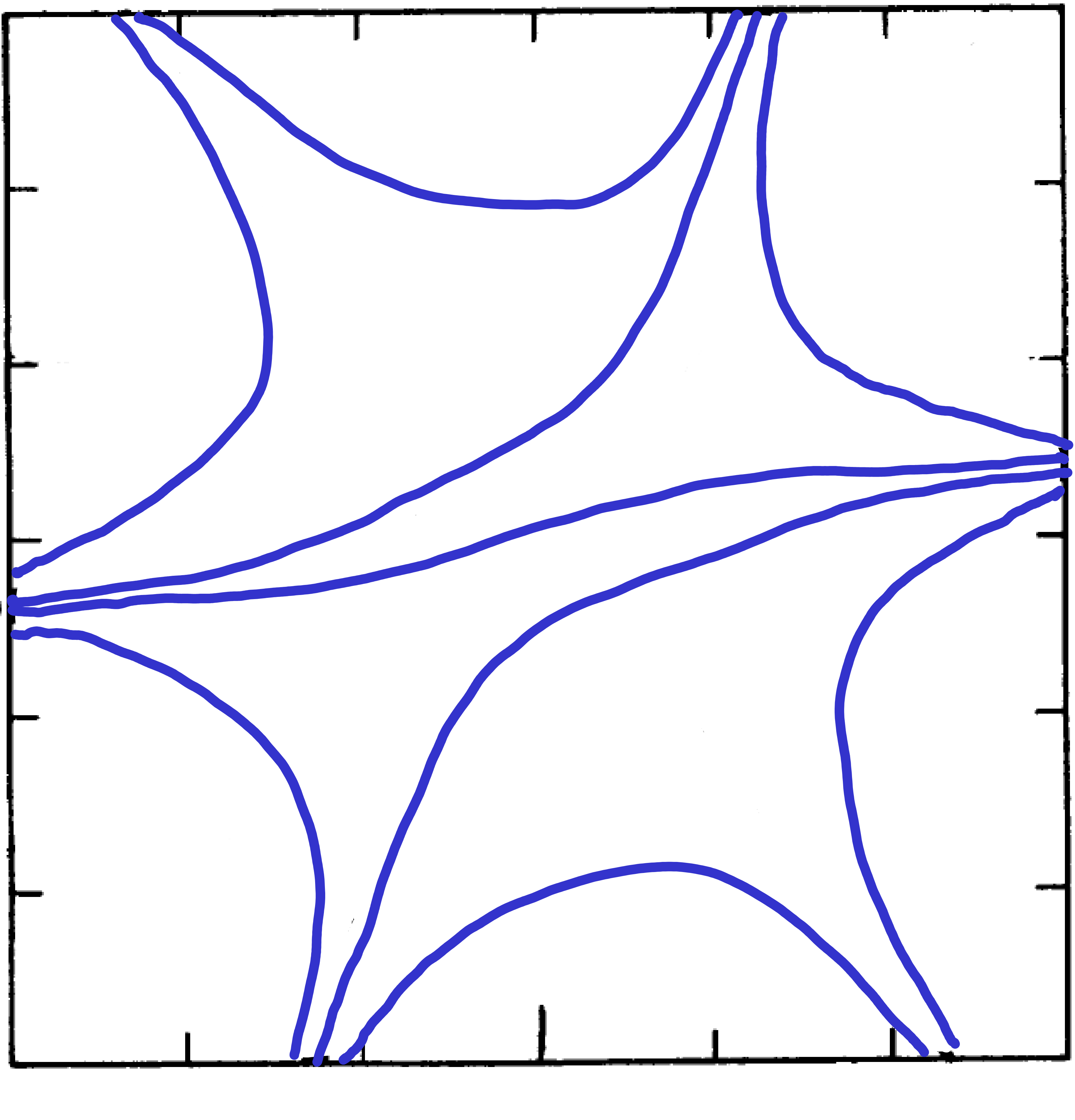
FIG. 19.

— Stokes lines.
∿ Cuts.









Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
6	4	3	2	2	2	2	2	2	2	2
1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

Tame ← | → Wild

affine Weyl group	minimal rank of bundles	pole orders
E_8	6	1+1+1
E_7	4	1+1+1
E_6	3	1+1+1
D_4	2	1+1+1+1
$A_3 = D_3$	2	2+1+1
D_2	2	2+2
D_1	2	2+2
D_0	2	2+2
A_2	2	3+1
A_1	2	4
A_0	2	4

affine Weyl group

minimal rank of bundles

pole orders

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

E_8 E_7 E_6

		P_3	P_3'	P_3''
		D_2	D_1	D_0
D_4	$A_3 = D_3$			
P_6	P_5			
		A_2	A_1	A_0
		P_4	P_2	P_1

Phase spaces for Painlevé differential equations

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

$\mathcal{M}^* \cong$ ALE

$\mathcal{M}^* \cong$ ALF

E_8 E_7 E_6

D_4 $A_3 = D_3$

D_2 D_1 D_0 Atiyah-Hitchin

A_2 A_1 A_0

$T^*\mathbb{P}^1$ $\mathbb{C}^2$

[$\mathcal{M}^* \subset \mathcal{M}$ open piece where bundle holom. trivial]