Quantum speed limit and displacement energy

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joint work with Leonid Polterovich

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1. Quantum speed limit
2. Displacement energy
3. Relationship in the semiclassical limit
Quantum speed limit

Consider a finite dimensional Hilbert space $\mathcal{H}$, a normalised state $\Psi \in \mathcal{H}$ and a quantum Hamiltonian $(\hat{H}_t) \in C^\infty([0, 1], \text{Herm}(\mathcal{H}))$. Solve Schrödinger equation

$$i\hbar \Psi'_t = \hat{H}_t \Psi_t$$

with initial condition $\Psi_0 = \Psi$.\n
Proposition (Quantum speed limit)

If $(\hat{H}_t)$ $a$-dislocates $\Psi$, then $
\ell_q(\hat{H}) \geq \hbar \arccos(a)$.\n
Quantum speed limit

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We say that $\left( \hat{H}_t \right)$ \textit{a-dislocates} the state $\Psi$ if $|\langle \psi_0, \psi_1 \rangle| \leq a$. 

Proposition (Quantum speed limit)

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Quantum speed limit

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Define the energy of \( (\hat{H}_t) \)

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\ell_q(\hat{H}_t) = \int_0^1 \| \hat{H}_t \|_{\text{op}} \, dt
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Define the energy of $(\hat{H}_t)$

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If $(\hat{H}_t)$ $a$-dislocates $\Psi$, then $\ell_q(\hat{H}) \geq \hbar \arccos(a)$. 
Displacement energy

Consider a symplectic manifold $M$ and a classical Hamiltonian $(H_t) \in C^\infty(M)$. Let $X_t$ be the corresponding vector field, $\omega(X_t, \cdot) = dH_t$ and $(\phi_t)$ be its flow.

Theorem
For any open set $\Omega$ of $M$, there exists $C > 0$ such that for any classical Hamiltonian $(H_t)$ displacing $\Omega$, $\ell_{cl}(H_t) \geq C$.

The largest $C$ is called the displacement energy.

This is due to Hofer, Viterbo, Polterovich, McDuff-Lalonde.
Displacement energy

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Define the energy of $(H_t)$

$$\ell_{\text{cl}}(H_t) = \int_0^1 \|H_t\|_\infty \, dt$$

We say that $(H_t)$ displaces a subset $S$ of $M$ if $\phi_1(S) \cap S = \emptyset$.

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Dislocation

$(\hat{H}_t)$ a-dislocates the state $\psi$ if $|\langle \psi_0, \psi_1 \rangle| \leq a$.

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For any open set $\Omega$ of $M$, there exists $C > 0$ such that for any classical Hamiltonian $(H_t)$ displacing $\Omega$, $\ell_{cl}(H_t) \geq C$. 

Assume that \((H\hbar, \hbar \in \Lambda)\) is a quantization of \(M\) and \(\text{Op}_\hbar : \mathcal{C}^\infty(M) \to \text{End} \mathcal{H}_\hbar, \quad \hbar \in \Lambda\) a convenient map quantizing observables.

Proposition

If \((H_t)\) displaces the microsupport of \(\Psi\), then \(\hat{H}_t = \text{Op}_\hbar (H_t)\)-dislocate \(\Psi\).
Microsupport displacement implies dislocation

Assume that \((\mathcal{H}_\hbar, \hbar \in \Lambda)\) is a quantization of \(M\) and

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\text{Op}_\hbar : C^\infty(M) \rightarrow \text{End} \mathcal{H}_\hbar, \quad \hbar \in \Lambda
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a convenient map quantizing observables.

Let \((\psi_\hbar \in \mathcal{H}_\hbar, \hbar \in \Lambda)\) be a normalised state. Define its microsupport by

\[
x \notin \text{MS}(\Psi) \iff \text{there exists } f \in C^\infty(M) \text{ such that } f(x) \neq 0 \text{ and } \text{Op}(f)\Psi = O(\hbar^\infty).
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x \notin \text{MS}(\Psi) \iff \text{there exists } f \in \mathcal{C}^\infty(M) \text{ such that } f(x) \neq 0 \text{ and } \text{Op}(f)\Psi = \mathcal{O}(\hbar^\infty).
\]

Proposition

If \((H_t)\) displaces the microsupport of \(\Psi\), then \(\hat{H}_t = \text{Op}_\hbar(H_t)\) \(\mathcal{O}(\hbar^\infty)\)-dislocate \(\Psi\).
Questions

We have seen that microsupport displacement implies dislocation.

1. Is the converse true?
   a. for “classical” quantum states, yes.
   b. for Lagrangian states, no.
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1. Is the converse true?
   a. for “classical” quantum states, yes.
   b. for Lagrangian states, no.

2. What about $\ell_q(\hat{H}_t)$?
   a. lower bound $\sim 1$
   b. same order as quantum speed limit
Semi-classical setting

Let $M$ be a complex compact manifold and $L \to M$ be a positive Hermitian holomorphic line bundle.

Then $\text{curv}(L) = \frac{1}{i} \omega$ where $\omega \in \Omega^2(M)$ is symplectic.

Define the Hilbert space

$$\mathcal{H}_{\hbar} = H^0(M, \mathcal{O}(L^k)), \quad \text{with } \hbar = 1/k$$

and the scalar product $\langle \psi, \psi' \rangle = \int_M (\psi, \psi') \mu$ where $\mu$ is the Liouville measure.
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For any $f \in C^\infty(M)$, define the Toeplitz operator

$$T_\hbar (f) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar, \quad T_\hbar (f) \Psi = \Pi_\hbar (f \Psi)$$

where $\Pi_\hbar$ is the orthogonal projector from $C^\infty(M, L^k)$ onto $\mathcal{H}_\hbar$. 
On the index of Toeplitz operators of several complex variables, (1978), Inventiones Mathematicae, 50.

with J. Sjöstrand, Sur la singularité des noyaux de Bergman et Szegö, (1975), Asterisque no 34-35.

“Classical” quantum state

Classical state $\tau$: Borel probability measure of $M$.
Quantum mixed state: positive endomorphism of $\mathcal{H}_\hbar$ with trace 1.
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Classical state $\tau$: Borel probability measure of $M$. Quantum mixed state: positive endomorphism of $\mathcal{H}_\hbar$ with trace 1.

For any classical state $\tau$, define the mixed state

$$Q_\hbar(\tau) = \int_M P_{x,\hbar} \, d\tau(x)$$

where $P_{x,\hbar}$ is the projector onto $\mathbb{C}e_{x,\hbar}$ and $e_{x,\hbar} \in \mathcal{H}_\hbar$ is the coherent state at $x$. 
“Classical” quantum state

Classical state $\tau$: Borel probability measure of $M$. Quantum mixed state: positive endomorphism of $\mathcal{H}_\hbar$ with trace 1. For any classical state $\tau$, define the mixed state

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We say that $(\hat{H}_t)$ $a$-dislocates the mixed state $\theta$ if $F(U_t\theta U_t^*, \theta) \leq a$ where $U_t = \text{quantum propagator, } i\hbar U'_t = \hat{H}_t U_t$. $F = \text{fidelity, } F(\theta, \sigma) = \|\theta^{1/2}\sigma^{1/2}\|_{\text{tr}}$

Another interesting fidelity is $F'(\theta, \sigma) = \left(\frac{\langle \theta, \sigma \rangle_{\text{HS}}}{\|\theta\|_{\text{HS}}\|\sigma\|_{\text{HS}}}\right)^{1/2}$.
Displacement vs dislocation for “classical” quantum state

Theorem (C-Polterovich)

Let $\tau$ be a classical state, $(H_t)$ a classical Hamiltonian and $\hat{H}_t = T_\hbar(H_t)$.

1. if $(H_t)$ displaces the support of $\tau$, then $(\hat{H}_t) \mathcal{O}(\hbar^\infty)$-dislocates $Q_\hbar(\tau)$.
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Let \( \tau \) be a classical state, \((H_t)\) a classical Hamiltonian and \( \hat{H}_t = T_\hbar(H_t) \).

1. if \((H_t)\) displaces the support of \( \tau \), then \((\hat{H}_t) \mathcal{O}(\hbar^{\infty})\)-dislocates \(Q_\hbar(\tau)\).

2. If \( \tau = f \mu \) with \( f \) of class \( \mathcal{C}^3 \) and \((\hat{H}_t) \circ(\hbar^n)\)-dislocate \(Q_\hbar(\tau)\), then for any \( \lambda > 0 \),
   
   \( (H_t) \) displaces \( \{f > \lambda\} \) when \( \hbar \) is sufficiently small
   
   \( \ell_q(H) \geq C + \mathcal{O}(\hbar) \) with \( C \) the displacement energy of \( \{f > \lambda\} \).
Theorem (C-Polterovich)

Let $\tau$ be a classical state of class $C^3$ such that

$$\mu(\text{Supp } \tau) < \mu(M)/2.$$

Then for any $\epsilon > 0$, there exists a classical Hamiltonian $(H_t)$ such that

$\quad \ell_{\text{cl}}(H_t) \leq \epsilon,$

$\quad \mu(\phi_1(\text{Supp } \tau) \cap \text{Supp } \tau) \leq \epsilon$ where $\phi_1$ is the time-one-map of the Hamiltonian flow of $(H_t)$,
Flexibility in displacement/dislocation

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Then for any $\epsilon > 0$, there exists a classical Hamiltonian $(H_t)$ such that

- $\ell_{\text{cl}}(H_t) \leq \epsilon$,
- $\mu(\phi_1(\text{Supp } \tau) \cap \text{Supp } \tau) \leq \epsilon$ where $\phi_1$ is the time-one-map of the Hamiltonian flow of $(H_t)$,
- the corresponding quantum Hamiltonian $(\hat{H}_t)$ $\epsilon$-dislocates $Q_{\hbar}(\tau)$ when $\hbar$ is sufficiently small and $\ell_q(\hat{H}) \leq \epsilon$. 

Dislocation of Lagrangian state

Let $\Gamma \subset M$ be a Lagrangian submanifold.

A Lagrangian state $(\psi_\hbar)$ supported by $\Gamma$ satisfies

1. $\psi_\hbar(x) = \mathcal{O}(\hbar^\infty)$ if $x \notin \Gamma$.
2. $\psi_\hbar(x) = \hbar^{-n/2} u^k(x) (a_0(x) + \hbar a_1(x) + \ldots)$ where $u(x) \in L_x$ has norm 1 and the $a_\ell(x)$’s are complex numbers.
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Let $\hat{H}_\hbar = \hbar T_\hbar(f(\cdot, \hbar))$ with $f(\cdot, \hbar) = f_0 + \hbar f_1 + \ldots$. Then

- $\ell_q(\hat{H}_\hbar) = \|\hat{H}_\hbar\|_{op} = \mathcal{O}(\hbar)$ and $\exp(i\hbar^{-1} \hat{H}_\hbar) = T_\hbar(e^{if_0}) + \mathcal{O}(\hbar)$.

- $\exp(i\hbar^{-1} \hat{H}_\hbar) \Psi_\hbar$ is still a Lagrangian state supported by $\Gamma$. 

Theorem (C-Polterovich)

Assume that $\Gamma = S^1 \times N$ and for any $x \in \Gamma$, $a_0(x) \neq 0$. Then we can choose the coefficients $f_\ell$’s so that $\hat{H}_\hbar$-dislocates $\Psi_\hbar$. 

Dislocation of Lagrangian state

Let $\Gamma \subset M$ be a Lagrangian submanifold.

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Let $\hat{H}_\hbar = \hbar T_\hbar(f(\cdot, \hbar))$ with $f(\cdot, \hbar) = f_0 + \hbar f_1 + \ldots$. Then

- $\ell_q(\hat{H}_\hbar) = \|\hat{H}_\hbar\|_{\text{op}} = O(\hbar)$ and $\exp(i\hbar^{-1}\hat{H}_\hbar) = T_\hbar(e^{i\tilde{f}_0}) + O(\hbar)$.
- $\exp(i\hbar^{-1}\hat{H}_\hbar)\Psi_\hbar$ is still a Lagrangian state supported by $\Gamma$.

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Assume that $\Gamma = S^1 \times N$ and for any $x \in \Gamma$, $a_0(x) \neq 0$. Then we can choose the coefficients $f_\ell$’s so that $\hat{H}_\hbar O(\hbar^{\infty})$-dislocates $\Psi_\hbar$. 