Quantum speed limit and displacement energy

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joint work with Leonid Polterovich

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Outline

1. Quantum speed limit
2. Displacement energy
3. Relationship in the semiclassical limit

Quantum speed limit

Consider a finite dimensional Hilbert space $\mathcal{H}$, a normalised state $\Psi \in \mathcal{H}$ and a quantum Hamiltonian $(\hat{H}_t) \in \mathcal{C}^\infty([0,1], \mathrm{Herm}(\mathcal{H}))$. Solve Schrödinger equation

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with initial condition $\Psi_0 = \Psi$.

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Proposition (Quantum speed limit)

If $(\hat{H}_t)$ a-dislocates Ψ, then $\ell_q(\hat{H}) \geqslant \hbar \arccos(a)$.

Displacement energy

Consider a symplectic manifold M and a classical Hamiltonian $(H_t) \in \mathcal{C}^\infty(M)$. Let X_t be the corresponding vector field, $\omega(X_t, \cdot) = dH_t$ and (ϕ_t) be its flow.

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Theorem

For any open set Ω of M, there exists $C > 0$ such that for any classical Hamiltonian (H_t) displacing Ω, $\ell_{\text{cl}}(H_t) \geqslant C$.

The largest C is called the displacement energy.
This is due to Hofer, Viterbo, Polterovich, McDuff-Lalonde.

Dislocation

$(\hat{H}_t)$ *a-dislocates* the state Ψ if $\left|\langle \Psi_0, \Psi_1 \rangle\right| \leqslant a$.

Proposition

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Microsupport displacement implies dislocation

Assume that $(\mathcal{H}_\hbar, \hbar \in \Lambda)$ is a quantization of M and

$$\mathrm{Op}_\hbar : \mathcal{C}^\infty(M) \to \mathrm{End}\,\mathcal{H}_\hbar, \qquad \hbar \in \Lambda$$

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Let $(\Psi_\hbar \in \mathcal{H}_\hbar, \hbar \in \Lambda)$ be a normalised state. Define its microsupport by

$$x \notin \mathrm{MS}(\Psi) \Leftrightarrow \text{there exists } f \in \mathcal{C}^\infty(M) \text{ such that } f(x) \neq 0 \text{ and } \mathrm{Op}(f)\Psi = \mathcal{O}(\hbar^\infty).$$

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Proposition

If (H_t) displaces the microsupport of Ψ, then $\hat{H}_t = \mathrm{Op}_\hbar(H_t)$ $\mathcal{O}(\hbar^\infty)$-dislocate Ψ.

Questions

We have seen that microsupport displacement implies dislocation.

1. Is the converse true ?
   a. for "classical" quantum states, yes.
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1. Is the converse true ?
   a. for "classical" quantum states, yes.
   b. for Lagrangian states, no.

2. What about $\ell_{\mathsf{q}}(\hat{H}_t)$?
   a. lower bound ~ 1
   b. same order as quantum speed limit

Semi-classical setting

Let M be a complex compact manifold and $L \to M$ be a positive Hermitian holomorphic line bundle.

Then $\mathrm{curv}(L) = \frac{1}{i}\omega$ where $\omega \in \Omega^2(M)$ is symplectic.

Define the Hilbert space

$$\mathcal{H}_\hbar = H^0(M, \mathcal{O}(L^k)), \qquad \text{with } \hbar = 1/k$$

and the scalar product $\langle \Psi, \Psi' \rangle = \int_M (\Psi, \Psi')\mu$ where μ is the Liouville measure.

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For any $f \in \mathcal{C}^\infty(M)$, define the Toeplitz operator

$$T_\hbar(f) : \mathcal{H}_\hbar \to \mathcal{H}_\hbar, \qquad T_\hbar(f)\Psi = \Pi_\hbar(f\Psi)$$

where $\Pi_\hbar$ is the orthogonal projector from $\mathcal{C}^\infty(M, L^k)$ onto $\mathcal{H}_\hbar$.

Louis Boutet de Monvel

On the index of Toeplitz operators of several complex variables, (1978), Inventiones Mathematicae, 50.

with J. Sjöstrand, *Sur la singularité des noyaux de Bergman et Szegö*, (1975), Asterisque no 34-35.

with V. Guillemin, *The spectral theory of Toeplitz operator*, (1981), Annals of Mathematics Studies, 99.

“Classical” quantum state

Classical state τ: Borel probability measure of M.
Quantum mixed state: positive endomorphism of $\mathcal{H}_\hbar$ with trace 1.

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For any classical state τ, define the mixed state

$$Q_\hbar(\tau) = \int_M P_{x,\hbar} \, d\tau(x)$$

where $P_{x,\hbar}$ is the projector onto $\mathbb{C}e_{x,\hbar}$ and $e_{x,\hbar} \in \mathcal{H}_\hbar$ is the coherent state at x.

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We say that $(\hat{H}_t)$ a-dislocates the mixed state θ if $F(U_t \theta U_t^*, \theta) \leqslant a$ where

- U_t = quantum propagator, $i\hbar U_t' = \hat{H}_t U_t$.
- F = fidelity, $F(\theta, \sigma) = \|\theta^{1/2}\sigma^{1/2}\|_{\mathrm{tr}}$

Another interesting fidelity is $F'(\theta, \sigma) = \left(\frac{\langle \theta, \sigma \rangle_{\mathrm{HS}}}{\|\theta\|_{\mathrm{HS}} \|\sigma\|_{\mathrm{HS}}} \right)^{1/2}$.

Displacement vs dislocation for "classical" quantum state

Theorem (C-Polterovich)

Let τ be a classical state, (H_t) a classical Hamiltonian and $\hat{H}_t = T_\hbar(H_t)$.

1. *if (H_t) displaces the support of τ, then $(\hat{H}_t)$ $\mathcal{O}(\hbar^\infty)$-dislocates $Q_\hbar(\tau)$.*

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1. *if (H_t) displaces the support of τ, then $(\hat{H}_t)$ $\mathcal{O}(\hbar^\infty)$-dislocates $Q_\hbar(\tau)$.*
2. *If $\tau = f\mu$ with f of class $\mathcal{C}^3$ and $(\hat{H}_t)$ $o(\hbar^n)$-dislocate $Q_\hbar(\tau)$, then for any $\lambda > 0$,*
   - *(H_t) displaces $\{f > \lambda\}$ when $\hbar$ is sufficiently small*
   - *$\ell_q(\hat{H}) \geqslant C + \mathcal{O}(\hbar)$ with C the displacement energy of $\{f > \lambda\}$.*

Flexibility in displacement/dislocation

Theorem (C-Polterovich)

Let τ be a classical state of class $\mathcal{C}^3$ such that

$$\mu(\operatorname{Supp}\tau) < \mu(M)/2.$$

Then for any $\epsilon > 0$, there exists a classical Hamiltonian (H_t) such that

- $\ell_{\mathrm{cl}}(H_t) \leqslant \epsilon$,
- $\mu(\phi_1(\operatorname{Supp}\tau) \cap \operatorname{Supp}\tau) \leqslant \epsilon$ *where ϕ_1 is the time-one-map of the Hamiltonian flow of (H_t),*

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- $\mu(\phi_1(\operatorname{Supp}\tau) \cap \operatorname{Supp}\tau) \leqslant \epsilon$ *where ϕ_1 is the time-one-map of the Hamiltonian flow of (H_t),*
- *the corresponding quantum Hamiltonian $(\hat{H}_t)$ ϵ-dislocates $Q_\hbar(\tau)$ when $\hbar$ is sufficiently small and $\ell_q(\hat{H}) \leqslant \epsilon$.*

Dislocation of Lagrangian state

Let $\Gamma \subset M$ be a Lagrangian submanifold.

A Lagrangian state $(\Psi_\hbar)$ supported by Γ satisfies

1. $\Psi_\hbar(x) = \mathcal{O}(\hbar^\infty)$ if $x \notin \Gamma$.
2. $\Psi_\hbar(x) = \hbar^{-n/2} u^k(x)(a_0(x) + \hbar a_1(x) + \ldots)$ where $u(x) \in L_x$ has norm 1 and the $a_\ell(x)$'s are complex numbers.

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Let $\hat{H}_\hbar = \hbar T_\hbar(f(\cdot, \hbar))$ with $f(\cdot, \hbar) = f_0 + \hbar f_1 + \ldots$. Then

- $\ell_q(\hat{H}_\hbar) = \|\hat{H}_\hbar\|_{\text{op}} = \mathcal{O}(\hbar)$ and $\exp(i\hbar^{-1}\hat{H}_\hbar) = T_\hbar(e^{if_0}) + \mathcal{O}(\hbar)$.
- $\exp(i\hbar^{-1}\hat{H}_\hbar)\Psi_\hbar$ is still a Lagrangian state supported by Γ.

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Theorem (C-Polterovich)

Assume that $\Gamma = S^1 \times N$ and for any $x \in \Gamma$, $a_0(x) \neq 0$. Then we can choose the coefficients f_ℓ's so that $\hat{H}_\hbar$ $\mathcal{O}(\hbar^\infty)$-dislocates $\Psi_\hbar$.