Almost global solutions for the periodic gravity-capillarity equation

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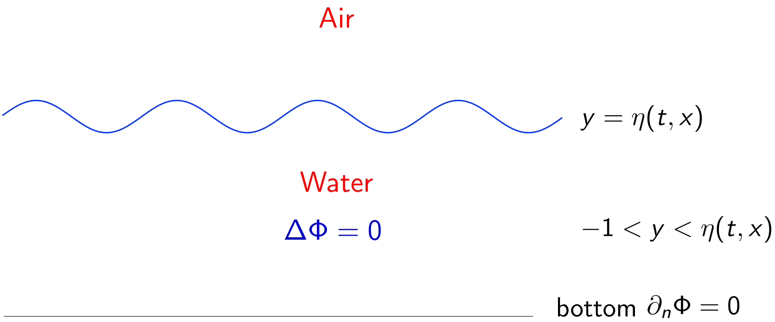
(joint work with Massimiliano Berti)

1. The gravity-capillarity wave equations



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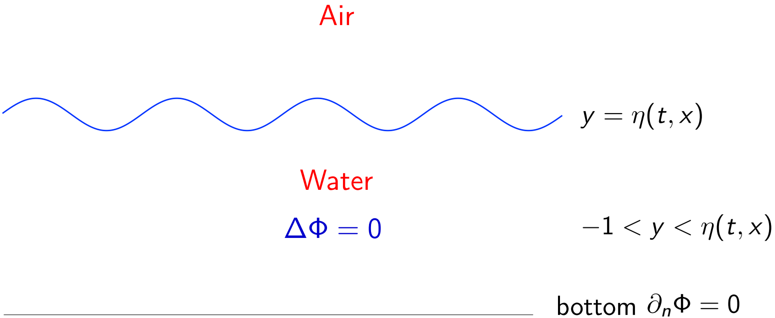

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Craig-Sulem-Zakharov formulation of the gravity-capillarity wave equations:

$$\text{(CGWE)}\quad \begin{aligned} \partial_t \eta &= G(\eta)\psi \\ \partial_t \psi &= -g\eta - \frac{1}{2}(\partial_x \psi)^2 + \frac{[G(\eta)\psi + \partial_x \psi \partial_x \eta]^2}{2(1+(\partial_x \eta)^2)} + \kappa H(\eta) \end{aligned}$$

where $G(\eta)\psi$ is the Dirichlet-Neumann operator defined by $G(\eta)\psi = (\partial_y \Phi - \partial_x \eta \partial_x \Phi)|_{y=\eta(t,x)}$ where $H(\eta) = \partial_x \left[\frac{\eta'}{\sqrt{1+\eta'^2}} \right]$, with $\eta' = \partial_x \eta$, where $g > 0, \kappa > 0$, and $x \in \mathbb{T}^1$.

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Known results:

• Local existence: Nalimov, Yoshihara, S. Wu, Lannes, Coutand-Shkroller, Beyer and Gunther, Ming-Zhang, Alazard-Burq-Zuily, Ambrose, Ambrose-Masmoudi, Schweizer.

• Long time existence with small decaying data:
Case $\kappa = 0$: Sijue Wu, Germain-Masmoudi-Shatah, Ionescu-Pusateri, Alazard-D., Ifrim-Tataru, Wang .
Case $\kappa > 0$: Deng-Ionescu-Pausader-Pusateri, Germain-Masmoudi-Shatah, Ionescu-Pusateri.

• Long time existence for non localized data: Existence of solutions with data of size ϵ over an interval of time of length ϵ^{-2}.
Ifrim-Tataru ($g = 0$ or $\kappa = 0$, periodic data).

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Question: Can we do better?

Definition: Let $S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. One says that a solution $\begin{bmatrix} \eta \\ \psi \end{bmatrix}$ of (CGWE) is *reversible* if and only for any t $\begin{bmatrix} \eta(-t) \\ \psi(-t) \end{bmatrix} = S \begin{bmatrix} \eta(t) \\ \psi(t) \end{bmatrix}$.

Note that this implies that $\psi(0) = 0$.

Conversely, $\psi(0) = 0$ implies that the solution is reversible, since if we write the equation $\begin{bmatrix} \dot{\eta} \\ \dot{\psi} \end{bmatrix} = F \begin{bmatrix} \eta \\ \psi \end{bmatrix}$, one has

$$SF \begin{bmatrix} \eta \\ \psi \end{bmatrix} = -F \left(S \begin{bmatrix} \eta \\ \psi \end{bmatrix} \right).$$

Notation: • $\eta \in H^{s+\frac{1}{4}}_{0,\mathrm{ev}}(\mathbb{T}^1)$ Sobolev space of even functions with zero mean.

• $\psi \in \dot{H}^{s-\frac{1}{4}}_{\mathrm{ev}}(\mathbb{T}^1)$ Sobolev space of even functions modulo constants.

Theorem

There is a zero measure subset $\mathcal{N}$ of $]0,+\infty[^2$, and for any (g,κ) in $]0,+\infty[^2-\mathcal{N}$, for any N in $\mathbb{N}$, there is $s_0>0$ and for any $s>s_0$, there are $c>0, \epsilon_0>0$ such that, for any $\epsilon<\epsilon_0$, any $\eta_0 \in H^{s+\frac{1}{4}}_{0,\mathrm{ev}}(\mathbb{T}^1)$, with norm smaller than ϵ, (CGWE) has a unique solution $(\eta,\psi) \in C^0(]-T_\epsilon, T_\epsilon[, H^{s+\frac{1}{4}}_{0,\mathrm{ev}}(\mathbb{T}^1) \times \dot{H}^{s-\frac{1}{4}}_{\mathrm{ev}}(\mathbb{T}^1))$ with $T_\epsilon \geq c\epsilon^{-N}$, and Cauchy data $(\eta,\psi)|_{t=0} = (\eta_0, 0)$.

Reference: There exist special global solutions: time periodic (Alazard-Baldi) or quasi-periodic (Berti-Montalto) solutions.

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2. Principle of proof on a model

Let $m \in \mathbb{R}_+^*$ and $\Lambda_m = \sqrt{-\Delta + m^2}$ acting on $L^2(\mathbb{T}^1)$. Let u be a solution to

$$(D_t - \Lambda_m)u = P(u, \bar{u})$$
$$u\big|_{t=0} = \epsilon u_0, \quad u_0 \in H^s(\mathbb{T}^1, \mathbb{C}) \; s \gg 1$$

where P is a polynomial homogeneous of degree p. The Sobolev energy inequality is

$$\frac{d}{dt}\|u(t,\cdot)\|_{H^s}^2 = \frac{d}{dt}\langle \Lambda_m^s u, \Lambda_m^s u\rangle_{L^2} = 2\mathrm{Re}\, i\langle \Lambda_m^s P(u,\bar{u}), \Lambda_m^s u\rangle$$

whence

$$\|u(t,\cdot)\|_{H^s} \leq \underbrace{\|u(0,\cdot)\|_{H^s}}_{\sim\epsilon} + C\int_0^t \|u(\tau,\cdot)\|_{H^s}^p \, d\tau.$$

One gets an a priori bound on an interval of length at least $c\epsilon^{-p+1}$, which implies existence up to such a time. One can get a better result by a normal forms method.

Look for some $Q(u,\bar u)$ homogeneous of degree p such that

$$(D_t - \Lambda_m)[u + Q(u,\bar u)] = (\text{terms of order } q > p)$$
$$+ (\text{terms of order } p \text{ that do not contribute to the energy}).$$

Then as $\|Q(u,\bar u)\|_{H^s} = O(\|u\|^2_{H^s})$, one gets a time of existence in $c\epsilon^{-q+1}$.

Take $P(u,\bar u) = u^\ell \bar u^{p-\ell}$ and look for $Q = M(\underbrace{u,\dots,u}_{\ell},\underbrace{\bar u,\dots,\bar u}_{p-\ell})$

such that $(D_t - \Lambda_m)Q = -u^\ell \bar u^{p-\ell} + \text{ h. o. terms.}$ Then

$$(D_t - \Lambda_m)M(u,\dots,u,\bar u,\dots,\bar u) = \sum_1^\ell M(u,\dots,\Lambda_m u,\dots,u,\bar u,\dots,\bar u)$$

$$-\sum_{\ell+1}^p M(u,\dots,u,\bar u,\dots,\Lambda_m \bar u,\dots,\bar u) - \Lambda_m M(u,\dots,\bar u) + \text{ h. o. terms.}$$

Denote by Π_n the spectral projector associated to the n-th mode of $-\Delta$ on $\mathbb{T}^1$. Replace the j-th argument u by $\Pi_{n_j} u_j$ and $\Lambda_m \Pi_{n_j} u_j$ by $\sqrt{m^2 + n_j^2}\,\Pi_{n_j} u_j$, and make act on the equation $\Pi_{n_{p+1}}$.

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One gets to solve

$$\mathcal{D}_\ell(n_1,\dots,n_{p+1})\Pi_{n_{p+1}}M(\Pi_{n_1}u_1,\dots,\Pi_{n_p}u_p) = -\Pi_{n_{p+1}}\Big(\prod_1^p \Pi_{n_j}u_j\Big)$$

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One proves that if m is outside a subset of zero measure and $\sum_1^{p+1} \pm n_j = 0$, then

$$|\mathcal{D}_\ell(n_1,\dots,n_{p+1})| \geq c(\text{third largest among } n_1,\dots,n_{p+1})^{-N_0}$$

except in the trivial case

$$p \text{ odd}, \ell = \frac{p+1}{2}, \{n_1,\dots,n_\ell\} = \{n_{\ell+1},\dots,n_{p+1}\}.$$

Using the *structure* of the equation, one may check that the corresponding terms do not contribute to the energy.

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Using the *structure* of the equation, one may check that the corresponding terms do not contribute to the energy.

3. Case of quasi-linear equations

Consider for instance $(D_t - \Lambda_m)u = |u|^2 D_x u$. The above procedure would generate a loss of one derivative in the estimates. Instead, one uses Bony's paralinearization formula
$uv = T_u v + T_v u + R(u,v)$ where

$$\widehat{T_u v} = \int_{|\xi-\eta|\ll|\eta|} \hat{u}(\xi-\eta)\hat{v}(\eta)\, d\eta$$

so that $T_u v$ has the same smoothness as v, and $R(u,v)$ is smoother than u, v. Our model may be written

$$(D_t - (\Lambda_m + T_{|u|^2} D_x))u = T_{D_x u}|u|^2 + R(u).$$

For (CGWE) we need to write the equation as a paradifferential system involving symbols that have a Taylor expansion in terms of the unknown (η, ψ) at an arbitrary order. The nonlinearity involves analytic expressions in $\partial_x \eta, \partial_x \psi$ and in $G(\eta)\psi$. One thus needs to express the Dirichlet-Neumann operator from such symbols.

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$$(D_t - \mathrm{Op}^{\mathrm{BW}}(\sqrt{m^2 + \xi^2} + |u|^2\xi))u = \text{semi-linear} + \text{smoothing terms.}$$

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One knows that if η is smooth, the Dirichlet-Neumann operator $G(\eta)$ corresponding to the Dirichlet problem in a strip $-1 \leq y \leq \eta(x)$ is a pseudo-differential operator. If η has limited regularity, it may be written as a paradifferential operator (Alazard-Métivier, Alazard-Burq-Zuily).

Here we need also an asymptotic expansion of the symbols in terms of (η, ψ) and also an expansion of the remainders. Instead of using a variational approach, we construct a Boutet de Monvel paradifferential parametrix for the Dirichlet problem.

This allows to write the equation as

$$\left(D_t - \mathrm{Op}^{\mathrm{BW}}(A(\eta, \psi; t, x, \xi))\right)\begin{bmatrix}\eta \\ \psi\end{bmatrix} = R(\eta, \psi)\begin{bmatrix}\eta \\ \psi\end{bmatrix}$$

where A is a matrix of symbols of para-differential operators (depending on η, ψ), $R(\eta, \psi)$ a smoothing operator, that gains ρ derivatives, and $\mathrm{Op}^{\mathrm{BW}}(\cdot)$ stands for Bony-Weyl quantization.

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4. Sketch of proof

1st step: Reductions (Alazard-Métivier, Alazard-Baldi, Berti-Montalto)

One may rewrite the equation in terms of a new unknown, and perform series of reductions that bring to a new equation, in terms of a complex new unknown $U = \left[\begin{smallmatrix} u \\ \bar{u} \end{smallmatrix} \right]$, and an auxiliary $\mathbb{C}^2$ valued function V, expressed from U, such that $\|U\|_{\dot{H}^s} \sim \|V\|_{\dot{H}^s}$. This new equation, may be expressed in terms of constant coefficients operators, namely operators $a(D_x) = \mathcal{F}^{-1} a(\xi) \mathcal{F}$, where the Fourier multiplier a is either

- $m_\kappa(\xi) = (\xi \tanh \xi)^{1/2} (1 + \kappa^2 \xi^2)^{1/2}$.
- $H(U; t, \xi)$ is a diagonal matrix of symbols of order one, with $\mathrm{Im}\, H(U; t, \xi)$ of order zero.

One gets

$$\Big(D_t - m_\kappa(D_x)(1+\underline{\zeta}(U;t))\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right] + H(U;t,D_x)\Big)V = R(U)V$$

where :

- $\underline{\zeta}(U;t)$ is a function of t, *independent of* x.
- $R(U)$ is a ρ-smoothing operator.

Moreover, these operators satisfy with $S = -\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$ the

- **Reality condition**: $\overline{H(U;t,D_x)V} = -SH(U;t,D_x)S\overline{V}$, that reflects that the initial system was real valued.
- **Parity preservation condition**: $H(U;t,D_x)$ preserve even functions.
- **Reversibility condition**: $S[H(U;t,D_x)V] = -H(SU;t,D_x)SV$

The preceding equation implies an energy inequality:

$$\|V(t,\cdot)\|_{\dot{H}^s} \leq \|V(0,\cdot)\|_{\dot{H}^s} + C\int_0^t \|\mathrm{Im}\, H(U;\tau,D_x)V(\tau,\cdot)\|_{\dot{H}^s}\, d\tau.$$

If we knew that $\mathrm{Im}\, H(U;t,\xi) = O(\|U\|^N_{\dot{H}^s})$ when $U \to 0$, we would get

$$\|V(t,\cdot)\|_{\dot{H}^s} \leq \underbrace{\|V(0,\cdot)\|_{\dot{H}^s}}_{\sim\epsilon} + C\int_0^t \|U(\tau,\cdot)\|^N_{\dot{H}^s}\|V(\tau,\cdot)\|_{\dot{H}^s}\, d\tau$$

which would imply an a priori bound $\|V(t,\cdot)\|_{\dot{H}^s} \leq K\epsilon$ if $t \leq c/\epsilon^N$. The long time existence result would follow from that.

2nd step: Normal forms. One eliminates by normal forms those contributions to the symbol $\mathrm{Im}\, H(U; t, \xi)$ homogeneous of degree smaller than N. One proceeds as in the model case, dividing by $\mathcal{D}_\ell(n_1, \ldots, n_p) = \sum_1^\ell m_\kappa(n_j) - \sum_{\ell+1}^p m_\kappa(n_j)$, $n_j \in \mathbb{N}^*$

Lemma: If κ is outside a convenient subset of zero measure, and if one is **not** in the case

(H) $\qquad p$ even, $\ell = p/2, \{n_1, \ldots, n_\ell\} = \{n_{\ell+1}, \ldots, n_p\}$

then

$$|\mathcal{D}_\ell(n_1, \ldots, n_p)| \geq c(n_1 + \cdots + n_p)^{-N_0}$$

for some c, N_0.

This allows to solve the equation $\mathcal{D}_\ell(\cdots)B_p(\cdots) = i\mathrm{Im}\, H_p(\cdots)$, except in the case (H). But then

$$\mathrm{Im}\, H_p(\Pi^+_{n_1} U, \ldots, \Pi^+_{n_\ell} U, \Pi^-_{n_{\ell+1}} U, \ldots, \Pi^-_{n_p} U; t, \xi) \equiv 0$$

as a consequence of the reality, parity preservation and reversibility conditions.

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