The transmission property

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1. Boutet's transmission condition

One of the early mathematical achievements of Louis Boutet de Monvel was to establish in 1966-71 a calculus of boundary value problems for classical pseudodifferential operators P satisfying the so-called transmission condition at the boundary of an open smooth subset Ω of $\mathbb{R}^n$ (or of an n-dimensional manifold Ω_1).

Recall some details:
Inspired from a Doklady announcement of Vishik and Eskin 1964, Boutet introduced in J.An.Math. '66 (submitted June '65, CRAS note Nov. '65):

Definition 1. *A classical ψdo P of order m with symbol $p \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$ is said to satisfy the* transmission condition *at $\partial\Omega$, when*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x,-\nu) = e^{\pi i(m-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x,\nu), \qquad (1)$$

for all indices; here $x \in \partial\Omega$ and ν denotes the interior normal at x.

Define the truncation of P to Ω as $P_+ = r^+ P e^+$, where r^+ denotes restriction to Ω and e^+ denotes extension by zero.

Theorem 2. *The transmission condition is necessary and sufficient in order that P_+ maps $C^\infty(\overline{\Omega})$ into $C^\infty(\overline{\Omega})$.*

In another J.An.Math. '66 paper, Boutet introduced Poisson operators, arising as $\varphi \mapsto r^+ P(\varphi(x') \otimes \delta(x_n))$ when $\Omega = \mathbb{R}^n_+$, for P having the transmission property.

The theory was elaborated further in Ann.Inst.Fourier '69 and Acta Math. '71, introducing operator systems:

$$\mathcal{A} = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{matrix} C^\infty(\overline{\Omega})^N \\ \times \\ C^\infty(\partial\Omega)^M \end{matrix} \to \begin{matrix} C^\infty(\overline{\Omega})^{N'} \\ \times \\ C^\infty(\partial\Omega)^{M'} \end{matrix}, \text{ where}$$

- K is a Poisson operator from $\partial\Omega$ to Ω,
- T is a trace operator from Ω to $\partial\Omega$,
- S is a ψdo on $\partial\Omega$,
- G is a singular Green operator, e.g. of type KT, also including the "leftover operators" $L(P, Q) = (PQ)_+ - P_+ Q_+$.

Along the way, more techniques were introduced, e.g. the Wiener-Hopf calculus worked out with Laguerre-type expansions in Acta '71, restricting to operators P of integer order. Here ν and $-\nu$ can be exchanged, whereby P^* also has the transmission property.

T and G are of *class* d, when they contain the first d standard traces $\{\gamma_0, \dots, \gamma_{d-1}\}$.
The mappings extend to Sobolev spaces: For $\mathcal{A}$ of order m, when $s + m > d - \frac{1}{2}$,

$$\mathcal{A}\colon H^{s+m}(\Omega)^N \times H^{s+m-\frac{1}{2}}(\partial\Omega)^M \to H^s(\Omega)^{N'} \times H^{s-\frac{1}{2}}(\partial\Omega)^{M'}.$$

The ψdbo calculus defines an "algebra" of operators, where the composition of two systems leads to a third one (when the matrix dimensions match). It allows operators of all orders, both positive and negative. In particular, when a system is *elliptic* of order m, then there exists a *parametrix* (an approximate inverse) of order $-m$, which also belongs to the calculus.

The calculus has been used very much through the years to solve problems, both for pseudodifferential and for differential operators (which are a special case). It has not only provided new results, but also a notational framework for elegant formulations when P is a differential operator.

2. The μ-transmission condition

Likewise inspired by Vishik and Eskin's work, Hörmander in Princeton prepared a course in the year 1965-66 on ψdo's, where he introduced a more general transmission condition (independently of Boutet's note):

Definition 3. *Let $\mu \in \mathbb{C}$. A classical ψdo of order $m \in \mathbb{C}$ is said to have the* μ-transmission property *at $\partial\Omega$ (for short: to be* of type μ*), when for all indices,*

$$\partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu). \qquad (2)$$

Boutet's transmission condition is the case of (2) with $\mu = 0 \pmod{\mathbb{Z}}$.

When $\mu \notin \mathbb{Z}$, the μ-transmission condition is necessary and sufficient for a different property than preservation of $C^\infty(\overline{\Omega})$.

Definition 4. *For* $\operatorname{Re}\mu > -1$, *define*

$$\mathcal{E}_\mu(\overline{\Omega}) = e^+(d^\mu C^\infty(\overline{\Omega})),$$

where $d(x)$ is a smooth positive extension into Ω of $\operatorname{dist}(x, \partial\Omega)$ *near $\partial\Omega$.*

For lower values of Re μ, define the spaces $\mathcal{E}_\mu(\overline{\Omega})$ successively so that $\mathcal{E}_{\mu-1}(\overline{\Omega})$ is the linear hull of the spaces $D\mathcal{E}_\mu(\overline{\Omega})$, where D varies over the first-order differential operators with C^∞-coefficients.

Hörmander showed (and included as Th. 18.2.18 in his book '85, with a different notation):

Theorem 5. *For a classical ψdo P, the μ-transmission property at $\partial\Omega$ is necessary and sufficient in order that*

$$u \in \mathcal{E}_\mu(\overline{\Omega}) \implies r^+ Pu \in C^\infty(\overline{\Omega}). \qquad (3)$$

His notes also dicuss the solvability of the so-called *homogeneous Dirichlet problem* for such operators in L_2-Sobolev spaces:

$$r^+ Pu = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}. \qquad (4)$$

In particular, when P in Theorem 5 is elliptic with a certain factorization property, then (3) is a bi-implication for the solutions of (4).

Boutet received the notes from Hörmander when he visited Princeton in 1967. He cited them in the Ann.Inst.Fourier paper '69 where he formulated the μ-transmission idea for analytic ψdo's, and derived the analogue of (3) in the analytic setting. The notes are also mentioned in his contribution to Ann.Math.Stud. 91, Princeton '78.

In 1980 I received the notes, but I did not read deeply into them then.

At the memorial lectures for Hörmander at the Nordic-European Congress in Lund June 2013, both Louis and I made remarks on these particular operators. A prominent example is the square root Laplacian $(-\Delta)^{\frac{1}{2}}$, a first-order ψdo. It satisfies the $\frac{1}{2}$-transmission property, thus $d(x)^{\frac{1}{2}}$ must enter in discussions of its boundary problems.

While preparing the memorial lecture, I began to type up the notes in TEX, to force myself to read every word. They partly have the character of a rough sketch, with typos and omissions.

Over the summer 2013 I worked on developing the methods further, to get results in L_p-Sobolev spaces, leading to conclusions also for Hölder spaces. In this process, I found that my paper in CPDE '90 on generalizations of Boutet's calculus to L_p-spaces and in particular on an improved version of order-reducing operators, allowed substantial simplifications of the arguments in the notes.

I had a good correspondence with Louis in October 2013 about these things, where he appreciated my analysis and pointed to more references — and asked me for a new copy of the notes that he had lost long ago.

The rest of the talk will describe the new stuff.

Fractional Laplacians $(-\Delta)^a$ on $\mathbb{R}^n$, $0 < a < 1$, have in recent years been in focus in probability, finance and nonlinear PDE, but pseudodifferential methods were not used at all, although $(-\Delta)^a$ is a ψdo. Instead, real methods for singular integral operators and potential theory were used, plus a trick of Caffarelli and Silvestre 2007 that views $(-\Delta)^a$ as a Dirichlet-to-Neumann operator for a degenerate differential operator problem in $n+1$ variables.

For subsets Ω of $\mathbb{R}^n$, one studies the homogeneous Dirichlet problem:

$$r^+(-\Delta)^a u = f \text{ in } \Omega, \quad \operatorname{supp} u \subset \overline{\Omega}.$$

By a variational construction, there is unique solvability for $f \in L_2(\Omega)$, with $u \in \dot{H}^a(\overline{\Omega})$ (functions in $H^a(\mathbb{R}^n)$ supported in $\overline{\Omega}$).

Not much was known concerning higher regularity of u. There were early results by Vishik, Eskin, Shamir in the 1960's, namely: $u \in \dot{H}^{2a}(\overline{\Omega})$ when $a < \frac{1}{2}$, and $u \in \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\Omega})$ when $a \geq \frac{1}{2}$.
Recent results had been shown by Ros-Oton and Serra (JMPA '14): $f \in L_\infty$ implies $u \in d(x)^a C^s(\overline{\Omega})$, small s.

From the pseudodifferential point of view, $(-\Delta)^a$ is a classical ψdo of order $2a$ with **even** symbol:

$$p \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi), \quad p_j(x,-\xi) = (-1)^j p_j(x,\xi).$$

Such operators have the a-**transmission property** at a smooth $\partial\Omega$, and Theorem 5 applies. We need some notation for other function spaces:

1) The *Sobolev spaces* (Bessel-potential spaces) are defined for $1 < p < \infty$ (with $\langle\xi\rangle = (|\xi|^2+1)^{\frac{1}{2}}$, $1/p' = 1 - 1/p$) by:

$$\begin{aligned} H_p^s(\mathbb{R}^n) &= \{u \in \mathcal{S}'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle\xi\rangle^s \hat{u}) \in L_p(\mathbb{R}^n)\}, \\ \overline{H}_p^s(\Omega) &= r^+ H_p^s(\mathbb{R}^n), \\ \dot{H}_p^s(\overline{\Omega}) &= \{u \in H_p^s(\mathbb{R}^n) \mid \operatorname{supp} u \subset \overline{\Omega}\}. \end{aligned}$$

$\overline{H}_p^s(\Omega)$ and $\dot{H}_{p'}^{-s}(\overline{\Omega})$ are dual spaces. (The notation $\overline{H}$, $\dot{H}$ stems from Hörmander's works.) When $p = 2$, omit p.

2) The *Hölder spaces* $C^{k,\sigma}(\overline{\Omega})$ where $k \in \mathbb{N}_0$, $0 < \sigma \le 1$, are also denoted $C^s(\overline{\Omega})$ with $s = k + \sigma$ when $\sigma < 1$. For $s \in \mathbb{N}_0$, $C^s(\overline{\Omega})$ is the usual space of continuously differentiable functions. Hölder-Zygmund-spaces $C_*^s(\overline{\Omega})$ generalize $C^s(\overline{\Omega})$ ($s \in \mathbb{R}_+ \setminus \mathbb{N}$) to all $s \in \mathbb{R}$ with good interpolation properties. (The spaces C_*^s are also known as the Besov spaces $B_{\infty,\infty}^s$.) Also here the notation $\overline{C}$ and $\dot{C}$ can be used.

An important idea in Hörmander's '65 notes was to introduce the a-transmission spaces (in the case $p = 2$ by a difficult definition). Note that

$$(-\Delta + 1)^a = \mathrm{Op}((\langle \xi' \rangle^2 + \xi_n^2)^a) = \mathrm{Op}((\langle \xi' \rangle - i\xi_n)^a)\,\mathrm{Op}((\langle \xi' \rangle + i\xi_n)^a).$$

Denote $\Xi_\pm^t = \mathrm{Op}((\langle \xi' \rangle \pm i\xi_n)^t)$ on $\mathbb{R}^n$. Here Ξ_+^t preserves support in $\overline{\mathbb{R}}_+^n$, and Ξ_-^t preserves support in $\overline{\mathbb{R}}_-^n$, where $\mathbb{R}_\pm^n = \{x \in \mathbb{R}^n \mid x_n \gtrless 0\}$. Then for all $s \in \mathbb{R}$,

$$\Xi_+^t \colon \dot{H}_p^s(\overline{\mathbb{R}}_+^n) \overset{\sim}{\to} \dot{H}_p^{s-t}(\overline{\mathbb{R}}_+^n), \quad r^+\Xi_-^t e^+ \colon \overline{H}_p^s(\mathbb{R}_+^n) \overset{\sim}{\to} \overline{H}_p^{s-t}(\mathbb{R}_+^n).$$

In fact, Ξ_+^t and $r^+\Xi_-^t e^+$ are adjoints. The inverses are Ξ_+^{-t} resp. $r^+\Xi_-^{-t}e^+$. Now define the a-**transmission spaces**:

$$H_p^{a(s)}(\overline{\mathbb{R}}_+^n) = \Xi_+^{-a} e^+ \overline{H}_p^{s-a}(\mathbb{R}_+^n), \text{ for } s - a > -1/p'.$$

Here $e^+\overline{H}_p^{s-a}(\mathbb{R}_+^n)$ generally has a jump at $x_n = 0$; it is mapped by Ξ_+^{-a} to a singularity of the type x_n^a. In fact, we can show:

$$H_p^{a(s)}(\overline{\mathbb{R}}_+^n) \begin{cases} = \dot{H}_p^s(\overline{\mathbb{R}}_+^n) \text{ if } -1/p' < s - a < 1/p, \\ \subset e^+ x_n^a \overline{H}_p^{s-a}(\mathbb{R}_+^n) + \dot{H}_p^s(\overline{\mathbb{R}}_+^n) \text{ if } s - a > 1/p, \end{cases}$$

with $\dot{H}_p^s(\overline{\mathbb{R}}_+^n)$ replaced by $\dot{H}_p^{s-\varepsilon}(\overline{\mathbb{R}}_+^n)$ if $s - a - 1/p \in \mathbb{N}$.

The $\Xi^t_\pm$ do not have all the symbol estimates required for ψdo's. But it is possible to find modifications that are true ψdo's with the needed support preserving properties, and they can be extended to the curved situation (G CPDE '90). These are families of ψdo's $\Lambda^{(t)}_\pm$ of order t with the properties, for all s:

$$\Lambda^{(t)}_+ \colon \dot{H}^s_p(\overline{\Omega}) \overset{\sim}{\to} \dot{H}^{s-t}_p(\overline{\Omega}),$$
$$r^+\Lambda^{(t)}_- e^+ \colon \overline{H}^s_p(\Omega) \overset{\sim}{\to} \overline{H}^{s-t}_p(\Omega),$$

with inverses $\Lambda^{(-t)}_+$ resp. $r^+\Lambda^{(-t)}_- e^+$, and $\Lambda^{(t)}_-$, $\Lambda^{(t)}_+$ being adjoints.
For $\overline{\Omega}$, the role of x_n is taken over by $d(x)$. Then we define (consistently)

$$H^{a(s)}_p(\overline{\Omega}) = \Lambda^{(-a)}_+ e^+ \overline{H}^{s-a}_p(\Omega) \begin{cases} = \dot{H}^s_p(\overline{\Omega}) \text{ if } -1/p' < s-a < 1/p, \\ \subset e^+ d^a \overline{H}^{s-a}_p(\Omega) + \dot{H}^{s\,(-\varepsilon)}_p(\overline{\Omega}) \text{ if } s-a > 1/p. \end{cases}$$

Here $\mathcal{E}_a(\overline{\Omega}) \subset H^{a(s)}_p(\overline{\Omega})$ densely for all s, and $\bigcap_s H^{a(s)}_p(\overline{\Omega}) = \mathcal{E}_a(\overline{\Omega})$.

The operators $\Lambda^{(t)}_+$ have the t-transmission property, and the operators $\Lambda^{(t)}_-$ have the 0-transmission property, Consider P, classical of order $2a$ and even, hence having the a-transmission property. The idea is now to introduce

$$Q = \Lambda_-^{(-a)} P \Lambda_+^{(-a)};$$

it is of order 0 with 0-transmission property relative to Ω, hence belongs to the Boutet de Monvel calculus!

When P moreover is elliptic avoiding a ray (e.g., strongly elliptic), one can show that the principal symbol q_0 of Q has a factorization at $\partial\Omega$ into two bounded factors, in local coordinates:

$$q_0(x', \xi', \xi_n) = q_0^-(x', \xi', \xi_n) q_0^+(x', \xi', \xi_n),$$

where $Op(q_0^\pm)$ preserve support in $\overline{\mathbb{R}}^n_\pm$.
This can be used to show that the truncation $Q_+ = r^+ Q e^+$ defines a Fredholm operator

$$Q_+ \colon \overline{H}_p^s(\Omega) \xrightarrow{\sim} \overline{H}_p^s(\Omega), \text{ all } s > -1/p'.$$

Arguing carefully with the operators $\Lambda_\pm^{(a)}$, $r^\pm$ and $e^\pm$, one then finds:

Theorem 6. *Let P be a classical ψdo of order $2a$ with even symbol, elliptic avoiding a ray. Let $s > a - 1/p'$. The homogeneous Dirichlet problem*

$$r^+ Pu = f, \quad \operatorname{supp} u \subset \overline{\Omega},$$

considered for $u \in \dot{H}_p^{a-1/p'+\varepsilon}(\overline{\Omega})$, satifies:

$$f \in \overline{H}_p^{s-2a}(\Omega) \implies u \in H_p^{a(s)}(\overline{\Omega}), \text{ the } a\text{-transmission space.}$$

Moreover, the mapping from u to f is Fredholm:

$$r^+ P \colon H_p^{a(s)}(\overline{\Omega}) \to \overline{H}_p^{s-2a}(\Omega).$$

There is a similar result with H_p^s-spaces replaced by Triebel-Lizorkin scales $F_{p,q}^s$ and Besov scales $B_{p,q}^s$; in particular the Hölder-Zygmund scale C_*^s. E.g., for $s > a$,

$$f \in C_*^{s-2a}(\overline{\Omega}) \implies u \in C_*^{a(s)}(\overline{\Omega}) \subset e^+ d^a C_*^{s-a}(\overline{\Omega}) + \dot{C}_*^s(\overline{\Omega})$$

(with $\dot{C}_*^s(\overline{\Omega})$ replaced by $\dot{C}_*^{s-\varepsilon}(\overline{\Omega})$ if $s - a \in \mathbb{N}$). Improves Ros-Oton and Serra's 2014 result in an optimal way, for smooth sets Ω.

4. Nonhomogeneous boundary conditions

Up to now we have studied the so-called homogeneous Dirichlet problem.

Question: Is there a nontrivial "Dirichlet boundary value" on $\partial\Omega$, such that the problem represents the case where that value is zero?

To give a simple explanation, consider the C^∞-situation. Recall that

$$\mathcal{E}_a(\overline{\Omega}) = e^+(d^a C^\infty(\overline{\Omega})),$$

dense in $H_p^{a(s)}(\overline{\Omega})$.
In local coordinates where Ω is replaced by $\mathbb{R}^n_+$, $d(x)$ is the coordinate x_n. When $u \in \mathcal{E}_a(\overline{\Omega})$ and we write $u = d^a v$ with $v \in e^+ C^\infty(\overline{\Omega})$, a Taylor expansion of v gives for $x_n > 0$, $d = x_n$,

$$\begin{aligned} u(x) &= d^a[v(x',0) + d\partial_n v(x',0) + \tfrac{1}{2} d^2 \partial_n^2 v(x',0) + \dots] \\ &= d^a \gamma_0(\tfrac{u}{d^a}) + d^{a+1}\gamma_1(\tfrac{u}{d^a}) + \tfrac{1}{2} d^{a+2}\gamma_2(\tfrac{u}{d^a}) + \dots . \end{aligned} \tag{5}$$

Note furthermore that if $u \in \mathcal{E}_{a-1}(\overline{\Omega})$, then analogously

$$u = d^{a-1}\gamma_0(\tfrac{u}{d^{a-1}}) + d^a \gamma_1(\tfrac{u}{d^{a-1}}) + \tfrac{1}{2} d^{a+1}\gamma_2(\tfrac{u}{d^{a-1}}) + \dots , \tag{6}$$

an expansion of the type (5) except for the term with coefficient d^{a-1}.

So $\mathcal{E}_{a-1}(\overline{\Omega})$ contains $\mathcal{E}_a(\overline{\Omega})$, differing just by having a nonvanishing term $d^{a-1}\varphi$ in the start. We conclude:

Lemma 7. *$\mathcal{E}_a(\overline{\Omega})$ is the subset of $\mathcal{E}_{a-1}(\overline{\Omega})$ for which $\gamma_0(\frac{u}{d^{a-1}})$ vanishes.*

This extends by density to the *a*-transmission spaces. There are continuous maps

$$\gamma_{a,j} : u \mapsto \gamma_j(\tfrac{u}{d^a}) \text{ from } H_p^{a(s)}(\overline{\Omega}) \text{ to } B_p^{s-a-j-1/p}(\partial\Omega),$$

when $s > a + j + 1/p$, and there holds:

Lemma 8. *$H_p^{a(s)}(\overline{\Omega})$ is the subset of $H_p^{(a-1)(s)}(\overline{\Omega})$ for which $\gamma_0(\frac{u}{d^{a-1}})$ vanishes.*

The value $\gamma_0(\frac{u}{d^{a-1}})$ is the generalized *Dirichlet boundary value!*
And we can show:

Theorem 9. *The nonhomogeneous Dirichlet problem with u sought in $H_p^{(a-1)(s)}(\overline{\Omega})$,*

$$\begin{cases} r^+Pu = f \text{ on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, \\ \gamma_0(\frac{u}{d^{a-1}}) = \varphi \text{ on } \partial\Omega, \end{cases}$$

is Fredholm solvable for $s > a - 1 + 1/p$ with data $f \in \overline{H}_p^{s-2a}(\Omega)$, $\varphi \in B_p^{s-a+1-1/p}(\partial\Omega)$.

Note here that when u is such that $\gamma_0(\frac{u}{d^{a-1}}) = \varphi \neq 0$ at $x_0 \in \partial\Omega$, then $u(x)$ blows up like d^{a-1} when $x \to x_0$. The solutions with nonzero Dirichlet data are "large" in this sense (also observed by Abatangelo '15).

We can also study Neumann problems on $H^{(a-1)(s)}(\overline{\Omega})$:

$$\begin{cases} r^+ Pu = f \text{ on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, \\ \gamma_1(\frac{u}{d^{a-1}}) = \psi \text{ on } \partial\Omega, \end{cases}$$

with $f \in \overline{H}_p^{s-2a}(\Omega)$, $\psi \in B_p^{s-a-1/p}(\partial\Omega)$.
There is Fredholm solvability at least when P is principally like $(-\Delta)^a$. (G in A&PDE '14)

One can check that when u is in the smaller space $H^{a(s)}(\overline{\Omega})$ (i.e., when the Dirichlet value $\gamma_0(\frac{u}{d^{a-1}})$ vanishes), then the Neumann value $\gamma_1(\frac{u}{d^{a-1}})$ equals $\gamma_0(\frac{u}{d^a})$.

The above boundary conditions are *local*. There exist other well-posed boundary conditions for $(-\Delta)^a$, of interest in probability theory, that are nonlocal.

5. Integration by parts

Finally, just a few words on integration by parts. When u is in the Dirichlet domain $H^{a(s)}(\overline{\Omega})$, then as noted, $\gamma_0(\frac{u}{d^a})$ plays the role of a Neumann boundary value. We can show (G JDE'16):

Theorem 10. *Let P be a classical ψdo of order $2a$ $(0 < a < 1)$ with even symbol, elliptic avoiding a ray. Then for $u, u' \in H^{a(s)}(\overline{\Omega})$, $s > a + \frac{1}{2}$,*

$$\int_\Omega Pu\,\partial_j \bar{u}'\,dx + \int_\Omega \partial_j u\,\overline{P^* u'}\,dx,$$
$$= \Gamma(a+1)^2 \int_{\partial\Omega} s_0(x)\nu_j(x)\gamma_0(\tfrac{u}{d^a})\gamma_0(\tfrac{\bar{u}'}{d^a})\,d\sigma + \int_\Omega [P, \partial_j] u\,\bar{u}'\,dx,$$

where $[P, \partial_j]$ is the commutator $P\partial_j - \partial_j P$. Here ν_j is the j'th component of the normal vector ν, and $s_0(x) = p_0(x, \nu(x))$.

It was shown first for $P = (-\Delta)^a$ by Ros-Oton and Serra ARMA '14, then in a joint work with Valdinoci '16 for translation-invariant selfadjoint positive homogeneous P, by integral operator methods. The new thing here is to allow variable coefficients, nonselfadjointness, lower-order terms, by ψdo methods — in particular finding the new term with $[P, \partial_j]$.

Such formulas are useful for nonexistence proofs in nonlinear problems.

There is a corollary with ∂_j replaced by a radial derivative $x \cdot \nabla$, and of course other integral terms. As a corollary of this, we can show a Pohozaev-type formula for semilinear problems:
Consider the *nonlinear Dirichlet problem*

$$r^+ Pu = f(u), \quad \operatorname{supp} u \subset \overline{\Omega}, \qquad (7)$$

where $f(s) \in C^{0,1}(\mathbb{R})$. Let $F(t) = \int_0^t f(s)\, ds$.

Corollary 11. *Let P be selfadjoint. Any bounded real solution u of* (7) *satisfies the Pohozaev formula:*

$$-2n \int_\Omega F(u)\, dx + n \int_\Omega f(u)\, u\, dx$$
$$= \Gamma(1+a)^2 \int_{\partial\Omega} (x \cdot \nu)\, s_0 \gamma_0 \big(\tfrac{u}{d^a}\big)^2\, d\sigma + \int_\Omega [P, x \cdot \nabla] u\, u\, dx.$$

If P is x-independent, the last term is replaced by $\int_\Omega \operatorname{Op}(\xi \cdot \nabla_\xi p) u\, u\, dx$.

Example. Applied to $P = (-\Delta + m^2)^a$ with $f(u) = \operatorname{sign} u |u|^r$, this implies that on sharshaped domains, there are no nontrivial solutions in the *critical* and *supercritical* cases, where $r \geq \frac{n+2a}{n-2a}$. This follows from a sign analysis of the entering terms.

Details, if there is time: When $p(\xi) = (|\xi|^2 + m^2)^a$,

$$\xi \cdot \nabla p(\xi) = 2a|\xi|^2(|\xi|^2 + m^2)^{a-1} = 2a(|\xi|^2 + m^2)^a - 2am^2(|\xi|^2 + m^2)^{a-1},$$

so $\operatorname{Op}(\xi \cdot \nabla p(\xi)) = 2aP - P_1$, where $P_1 = 2am^2(-\Delta + m^2)^{a-1}$ is a positive operator. The Pohozaev identity is then

$$-2n \int_\Omega F(u)\, dx + (n-2a) \int_\Omega f(u)\, u\, dx + \int_\Omega P_1 u\, u\, dx \\ = \Gamma(1+a)^2 \int_{\partial\Omega} (x \cdot \nu)\, s_0\gamma_0\big(\tfrac{u}{d^a}\big)^2\, d\sigma.$$

When $f(u) = \operatorname{sign} u\, |u|^r$ with an $r > 1$, $F(u) = \frac{1}{r+1}|u|^{r+1}$, giving the formula

$$\frac{-2n+(n-2a)(r+1)}{r+1} \int_\Omega |u|^{r+1}\, dx + \int_\Omega P_1 u\, u\, dx = \Gamma(1+a)^2 \int_{\partial\Omega} (x \cdot \nu)\, s_0\gamma_0\big(\tfrac{u}{d^a}\big)^2\, d\sigma.$$

When $r \geq \frac{n+2a}{n-2a}$, the left-hand side is positive unless $u \equiv 0$. For star-shaped Ω, the right-hand side is ≤ 0. In such cases there are no nontrivial solutions.