The transmission property

Gerd Grubb
Copenhagen University

Boutet Memorial Conference
École Normale Supérieure, Paris
20–24 juin, 2016
1. Boutet’s transmission condition

One of the early mathematical achievements of Louis Boutet de Monvel was to establish in 1966-71 a calculus of boundary value problems for classical pseudodifferential operators $P$ satisfying the so-called transmission condition at the boundary of an open smooth subset $\Omega$ of $\mathbb{R}^n$ (or of an $n$-dimensional manifold $\Omega_1$).

Recall some details:

**Definition 1.** A classical $\psi$do $P$ of order $m$ with symbol $p \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi)$ is said to satisfy the transmission condition at $\partial \Omega$, when

$$\partial_\beta x \partial_\alpha \xi p_j(x, -\nu) = e^{\pi i (m-j-|\alpha|)} \partial_\beta x \partial_\alpha \xi p_j(x, \nu),$$

(1)

for all indices; here $x \in \partial \Omega$ and $\nu$ denotes the interior normal at $x$.

Define the truncation of $P$ to $\Omega$ as $P_+ = r^+ Pe^+$, where $r^+$ denotes restriction to $\Omega$ and $e^+$ denotes extension by zero.

**Theorem 2.** The transmission condition is necessary and sufficient in order that $P_+$ maps $C^\infty(\Omega)$ into $C^\infty(\overline{\Omega})$. 

Gerd Grubb Copenhagen University
In another J.An.Math. ’66 paper, Boutet introduced Poisson operators, arising as \( \varphi \mapsto r^+ P(\varphi(x') \otimes \delta(x_n)) \) when \( \Omega = \mathbb{R}_+^n \), for \( P \) having the transmission property.

The theory was elaborated further in Ann.Inst.Fourier ’69 and Acta Math. ’71, introducing operator systems:

\[
A = \begin{pmatrix} P_+ + G & K \\ T & S \end{pmatrix} : \begin{array}{c} C^\infty(\overline{\Omega})^N \\ \times \end{array} \begin{array}{c} C^\infty(\overline{\Omega})^{N'} \\ \times \end{array} \begin{array}{c} C^\infty(\partial\Omega)^M \\ \times \end{array} \begin{array}{c} C^\infty(\partial\Omega)^{M'} \end{array},
\]

where

- \( K \) is a Poisson operator from \( \partial\Omega \) to \( \Omega \),
- \( T \) is a trace operator from \( \Omega \) to \( \partial\Omega \),
- \( S \) is a \( \psi \)-do on \( \partial\Omega \),
- \( G \) is a singular Green operator, e.g. of type \( KT \), also including the “leftover operators” \( L(P, Q) = (PQ)_+ - P_+Q_+ \).

Along the way, more techniques were introduced, e.g. the Wiener-Hopf calculus worked out with Laguerre-type expansions in Acta ’71, restricting to operators \( P \) of integer order. Here \( \nu \) and \( -\nu \) can be exchanged, whereby \( P^\ast \) also has the transmission property.
$T$ and $G$ are of class $d$, when they contain the first $d$ standard traces \( \{\gamma_0, \ldots, \gamma_{d-1}\} \).

The mappings extend to Sobolev spaces: For $A$ of order $m$, when $s + m > d - \frac{1}{2}$,

$$A : H^{s+m}(\Omega)^N \times H^{s+m-\frac{1}{2}}(\partial\Omega)^M \to H^s(\Omega)^N' \times H^{s-\frac{1}{2}}(\partial\Omega)^M'.$$

The $\psi$dbo calculus defines an “algebra” of operators, where the composition of two systems leads to a third one (when the matrix dimensions match). It allows operators of all orders, both positive and negative. In particular, when a system is elliptic of order $m$, then there exists a parametrix (an approximate inverse) of order $-m$, which also belongs to the calculus.

The calculus has been used very much through the years to solve problems, both for pseudodifferential and for differential operators (which are a special case). It has not only provided new results, but also a notational framework for elegant formulations when $P$ is a differential operator.
Likewise inspired by Vishik and Eskin’s work, Hörmander in Princeton prepared a course in the year 1965-66 on \( \psi \)-do’s, where he introduced a more general transmission condition (independently of Boutet’s note):

**Definition 3.** Let \( \mu \in \mathbb{C} \). A classical \( \psi \)-do of order \( m \in \mathbb{C} \) is said to have the \( \mu \)-transmission property at \( \partial \Omega \) (for short: to be of type \( \mu \)), when for all indices,

\[
\partial_x^\beta \partial_\xi^\alpha p_j(x, -\nu) = e^{\pi i (m - 2\mu - j - |\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x, \nu).
\] (2)

Boutet’s transmission condition is the case of (2) with \( \mu = 0 \) (mod \( \mathbb{Z} \)).

When \( \mu \notin \mathbb{Z} \), the \( \mu \)-transmission condition is necessary and sufficient for a different property than preservation of \( C^\infty(\Omega) \).

**Definition 4.** For \( \text{Re} \mu > -1 \), define

\[
\mathcal{E}_\mu(\Omega) = e^+(d^\mu C^\infty(\Omega)),
\]

where \( d(x) \) is a smooth positive extension into \( \Omega \) of \( \text{dist}(x, \partial \Omega) \) near \( \partial \Omega \).
For lower values of $\text{Re} \mu$, define the spaces $\mathcal{E}_\mu(\overline{\Omega})$ successively so that $\mathcal{E}_{\mu-1}(\overline{\Omega})$ is the linear hull of the spaces $D\mathcal{E}_\mu(\overline{\Omega})$, where $D$ varies over the first-order differential operators with $C^\infty$-coefficients.

Hörmander showed (and included as Th. 18.2.18 in his book ’85, with a different notation):

**Theorem 5.** For a classical $\psi$do $P$, the $\mu$-transmission property at $\partial \Omega$ is necessary and sufficient in order that

$$u \in \mathcal{E}_\mu(\overline{\Omega}) \implies r^+ Pu \in C^\infty(\overline{\Omega}).$$

(3)

His notes also discuss the solvability of the so-called *homogeneous Dirichlet problem* for such operators in $L_2$-Sobolev spaces:

$$r^+ Pu = f \text{ in } \Omega, \quad \text{supp } u \subset \overline{\Omega}.$$ (4)

In particular, when $P$ in Theorem 5 is elliptic with a certain factorization property, then (3) is a bi-implication for the solutions of (4).

Boutet received the notes from Hörmander when he visited Princeton in 1967. He cited them in the Ann.Inst.Fourier paper ’69 where he formulated the $\mu$-transmission idea for analytic $\psi$do’s, and derived the analogue of (3) in the analytic setting. The notes are also mentioned in his contribution to Ann.Math.Stud. 91, Princeton ’78.
In 1980 I received the notes, but I did not read deeply into them then. At the memorial lectures for Hörmander at the Nordic-European Congress in Lund June 2013, both Louis and I made remarks on these particular operators. A prominent example is the square root Laplacian \((-\Delta)^{1/2}\), a first-order \(\psi\)do. It satisfies the \(1/2\)-transmission property, thus \(d(x)^{1/2}\) must enter in discussions of its boundary problems.

While preparing the memorial lecture, I began to type up the notes in \(\TeX\), to force myself to read every word. They partly have the character of a rough sketch, with typos and omissions.

Over the summer 2013 I worked on developing the methods further, to get results in \(L_p\)-Sobolev spaces, leading to conclusions also for Hölder spaces. In this process, I found that my paper in CPDE ’90 on generalizations of Boutet’s calculus to \(L_p\)-spaces and in particular on an improved version of order-reducing operators, allowed substantial simplifications of the arguments in the notes.

I had a good correspondence with Louis in October 2013 about these things, where he appreciated my analysis and pointed to more references — and asked me for a new copy of the notes that he had lost long ago.

The rest of the talk will describe the new stuff.
Fractional Laplacians \((-\Delta)^a\) on \(\mathbb{R}^n\), \(0 < a < 1\), have in recent years been in focus in probability, finance and nonlinear PDE, but pseudodifferential methods were not used at all, although \((-\Delta)^a\) is a \(\psi\) do. Instead, real methods for singular integral operators and potential theory were used, plus a trick of Caffarelli and Silvestre 2007 that views \((-\Delta)^a\) as a Dirichlet-to-Neumann operator for a degenerate differential operator problem in \(n + 1\) variables.

For subsets \(\Omega\) of \(\mathbb{R}^n\), one studies the homogeneous Dirichlet problem:

\[
\begin{align*}
  r^+ (-\Delta)^a u &= f \text{ in } \Omega, \\
  \text{supp } u &\subset \overline{\Omega}.
\end{align*}
\]

By a variational construction, there is unique solvability for \(f \in L^2(\Omega)\), with \(u \in \dot{H}^a(\overline{\Omega})\) (functions in \(H^a(\mathbb{R}^n)\) supported in \(\overline{\Omega}\)).

Not much was known concerning higher regularity of \(u\). There were early results by Vishik, Eskin, Shamir in the 1960’s, namely: \(u \in \dot{H}^{2a}(\overline{\Omega})\) when \(a < \frac{1}{2}\), and \(u \in \dot{H}^{a+\frac{1}{2}-\varepsilon}(\overline{\Omega})\) when \(a \geq \frac{1}{2}\).

Recent results had been shown by Ros-Oton and Serra (JMPA ’14): \(f \in L_\infty\) implies \(u \in d(x)^a C^s(\overline{\Omega})\), small \(s\).
From the pseudodifferential point of view, \((-\Delta)^a\) is a classical \(\psi\)do of order \(2a\) with even symbol:

\[ p \sim \sum_{j \in \mathbb{N}_0} p_j(x, \xi), \quad p_j(x, -\xi) = (-1)^j p_j(x, \xi). \]

Such operators have the \textbf{a-transmission property} at a smooth \(\partial \Omega\), and Theorem 5 applies. We need some notation for other function spaces:

1) The \textit{Sobolev spaces} (Bessel-potential spaces) are defined for \(1 < p < \infty\) (with \(\langle \xi \rangle = (|\xi|^2 + 1)^{\frac{1}{2}}\), \(1/p' = 1 - 1/p\)) by:

\[
H^s_p(\mathbb{R}^n) = \{ u \in S'(\mathbb{R}^n) \mid \mathcal{F}^{-1}(\langle \xi \rangle^s \hat{u}) \in L_p(\mathbb{R}^n) \}, \\
\overline{H}^s_p(\Omega) = r^+ H^s_p(\mathbb{R}^n), \\
\dot{H}^s_p(\Omega) = \{ u \in H^s_p(\mathbb{R}^n) \mid \text{supp } u \subset \Omega \}.
\]

\(\overline{H}^s_p(\Omega)\) and \(\dot{H}^{-s}_p(\Omega)\) are dual spaces. (The notation \(\overline{H}, \dot{H}\) stems from Hörmander’s works.) When \(p = 2\), omit \(p\).

2) The \textit{Hölder spaces} \(C^{k,\sigma}(\overline{\Omega})\) where \(k \in \mathbb{N}_0\), \(0 < \sigma \leq 1\), are also denoted \(C^s(\overline{\Omega})\) with \(s = k + \sigma\) when \(\sigma < 1\). For \(s \in \mathbb{N}_0\), \(C^s(\overline{\Omega})\) is the usual space of continuously differentiable functions. Hölder-Zygmund-spaces \(C^s_*(\overline{\Omega})\) generalize \(C^s(\overline{\Omega})\) \((s \in \mathbb{R}_+ \setminus \mathbb{N})\) to all \(s \in \mathbb{R}\) with good interpolation properties. (The spaces \(C^s_*\) are also known as the Besov spaces \(B^s_{\infty,\infty}\).)

Also here the notation \(\overline{C}\) and \(\dot{C}\) can be used.
An important idea in Hörmander’s ’65 notes was to introduce the $a$-transmission spaces (in the case $p = 2$ by a difficult definition). Note that

\[ (-\Delta + 1)^a = \text{Op}((\langle \xi' \rangle^2 + \xi_n^2)^a) = \text{Op}((\langle \xi' \rangle - i \xi_n)^a) \text{Op}((\langle \xi' \rangle + i \xi_n)^a). \]

Denote $\Xi_t^\pm = \text{Op}((\langle \xi' \rangle \pm i \xi_n)^t)$ on $\mathbb{R}^n$. Here $\Xi_t^+$ preserves support in $\mathbb{R}_+^n$, and $\Xi_t^-$ preserves support in $\mathbb{R}_-^n$, where $\mathbb{R}_+^n = \{ x \in \mathbb{R}^n \mid x_n \geq 0 \}$.

Then for all $s \in \mathbb{R}$,

\[ \Xi_t^+: \dot{H}^s_p(\mathbb{R}_+^n) \xrightarrow{\sim} \dot{H}^{s-t}_p(\mathbb{R}_+^n), \quad r^+\Xi_t^-e^+: \mathcal{H}^s_p(\mathbb{R}_+^n) \xrightarrow{\sim} \mathcal{H}^{s-t}_p(\mathbb{R}_+^n). \]

In fact, $\Xi_t^+$ and $r^+\Xi_t^-e^+$ are adjoints. The inverses are $\Xi_t^{-t}$ resp. $r^+\Xi_t^-e^+$. Now define the $a$-transmission spaces:

\[ H^a_p(s)(\mathbb{R}_+^n) = \Xi_{-a}^-e^+\mathcal{H}^{s-a}_p(\mathbb{R}_+^n), \text{ for } s - a > -1/p'. \]

Here $e^+\mathcal{H}^{s-a}_p(\mathbb{R}_+^n)$ generally has a jump at $x_n = 0$; it is mapped by $\Xi_{-a}^+$ to a singularity of the type $x_n^a$. In fact, we can show:

\[ H^a_p(s)(\mathbb{R}_+^n) \begin{cases} = \dot{H}^s_p(\mathbb{R}_+^n) \text{ if } -1/p' < s - a < 1/p, \\ \subset e^+x_n^a\mathcal{H}^{s-a}_p(\mathbb{R}_+^n) + \dot{H}^s_p(\mathbb{R}_+^n) \text{ if } s - a > 1/p, \end{cases} \]

with $\dot{H}^s_p(\mathbb{R}_+^n)$ replaced by $\dot{H}^{s-\varepsilon}_p(\mathbb{R}_+^n)$ if $s - a - 1/p \in \mathbb{N}$. 

Gerd Grubb Copenhagen University

The transmission property
The $\Xi_{\pm}$ do not have all the symbol estimates required for $\psi$do’s. But it is possible to find modifications that are true $\psi$do’s with the needed support preserving properties, and they can be extended to the curved situation (G CPDE '90). These are families of $\psi$do’s $\Lambda^{(t)}$ of order $t$ with the properties, for all $s$:

$$\Lambda^{(t)}_+: \mathring{H}_p^s(\Omega) \rightarrow \mathring{H}_p^{s-t}(\Omega),$$

$$r^+\Lambda^{(t)}_+e^+: \overline{H}_p^s(\Omega) \rightarrow \overline{H}_p^{s-t}(\Omega),$$

with inverses $\Lambda^{(-t)}_+$ resp. $r^+\Lambda^{(-t)}_-e^+$, and $\Lambda^{(t)}_-, \Lambda^{(t)}_+$ being adjoints.

For $\overline{\Omega}$, the role of $x_n$ is taken over by $d(x)$. Then we define (consistently)

$$H_p^{a(s)}(\overline{\Omega}) = \Lambda^{(-a)}_+e^+\overline{H}_p^{s-a}(\Omega)\left\{\begin{array}{ll}
= \mathring{H}_p^s(\overline{\Omega}) & \text{if } -1/p' < s - a < 1/p, \\
\subset e^+d^a\overline{H}_p^{s-a}(\Omega) + \mathring{H}_p^{s(-\varepsilon)}(\overline{\Omega}) & \text{if } s - a > 1/p.
\end{array}\right.$$  

Here $\mathcal{E}_a(\overline{\Omega}) \subset H_p^{a(s)}(\overline{\Omega})$ densely for all $s$, and $\bigcap_s H_p^{a(s)}(\overline{\Omega}) = \mathcal{E}_a(\overline{\Omega})$.

The operators $\Lambda^{(t)}_+$ have the $t$-transmission property, and the operators $\Lambda^{(t)}_-$ have the 0-transmission property,
Consider $P$, classical of order $2a$ and even, hence having the $a$-transmission property. The idea is now to introduce

$$Q = \Lambda^{(-a)}_- P \Lambda^{(-a)}_+;$$

it is of order 0 with 0-transmission property relative to $\Omega$, hence belongs to the Boutet de Monvel calculus!

When $P$ moreover is elliptic avoiding a ray (e.g., strongly elliptic), one can show that the principal symbol $q_0$ of $Q$ has a factorization at $\partial \Omega$ into two bounded factors, in local coordinates:

$$q_0(x', \xi', \xi_n) = q^-_0(x', \xi', \xi_n)q^+_0(x', \xi', \xi_n),$$

where $Op(q^0_\pm)$ preserve support in $\overline{\mathbb{R}}^n$. This can be used to show that the truncation $Q_+ = r^+ Q e^+$ defines a Fredholm operator

$$Q_+: \overline{H}^s_p(\Omega) \sim \overline{H}^s_p(\Omega), \text{ all } s > -1/p'.$$

Arguing carefully with the operators $\Lambda^{(a)}_\pm$, $r^\pm$ and $e^\pm$, one then finds:
Theorem 6. Let $P$ be a classical $\psi$do of order $2a$ with even symbol, elliptic avoiding a ray. Let $s > a - 1/p'$. The homogeneous Dirichlet problem

$$r^+ Pu = f, \quad \text{supp } u \subset \Omega,$$

considered for $u \in \dot{H}^{a-1/p' + \epsilon} (\Omega)$, satisfies:

$$f \in \overline{H}^{s-2a}_p (\Omega) \implies u \in H^a_p (\Omega), \text{ the a-transmission space}.$$

Moreover, the mapping from $u$ to $f$ is Fredholm:

$$r^+ P : H^a_p (\Omega) \rightarrow \overline{H}^{s-2a}_p (\Omega).$$

There is a similar result with $H^s_p$-spaces replaced by Triebel-Lizorkin scales $F^s_{p,q}$ and Besov scales $B^s_{p,q}$; in particular the Hölder-Zygmund scale $C^s_*$. E.g., for $s > a$,

$$f \in C^s_* (\Omega) \implies u \in C^a_* (\Omega) \subset e^+ d^a C^{s-a}_* (\Omega) + \dot{C}^s_* (\Omega)$$

(with $\dot{C}^s_* (\Omega)$ replaced by $\dot{C}^{s-\epsilon}_* (\Omega)$ if $s - a \in \mathbb{N}$). Improves Ros-Oton and Serra’s 2014 result in an optimal way, for smooth sets $\Omega$. 

Gerd Grubb Copenhagen University The transmission property
4. Nonhomogeneous boundary conditions

Up to now we have studied the so-called homogeneous Dirichlet problem. **Question:** Is there a nontrivial “Dirichlet boundary value” on $\partial \Omega$, such that the problem represents the case where that value is zero?

To give a simple explanation, consider the $C^\infty$-situation. Recall that

$$E_a(\Omega) = e^+(d^a C^\infty(\Omega)),$$

dense in $H^s_p(\Omega)$.

In local coordinates where $\Omega$ is replaced by $\mathbb{R}^n_+$, $d(x)$ is the coordinate $x_n$. When $u \in E_a(\Omega)$ and we write $u = d^a v$ with $v \in e^+ C^\infty(\Omega)$, a Taylor expansion of $v$ gives for $x_n > 0$, $d = x_n$,

$$u(x) = d^a [v(x', 0) + d\partial_n v(x', 0) + \frac{1}{2} d^2 \partial^2_n v(x', 0) + \ldots]$$

$$= d^a \gamma_0 \left( \frac{u}{d^a} \right) + d^{a+1} \gamma_1 \left( \frac{u}{d^a} \right) + \frac{1}{2} d^{a+2} \gamma_2 \left( \frac{u}{d^a} \right) + \ldots. \quad (5)$$

Note furthermore that if $u \in E_{a-1}(\Omega)$, then analogously

$$u = d^{a-1} \gamma_0 \left( \frac{u}{d^{a-1}} \right) + d^a \gamma_1 \left( \frac{u}{d^{a-1}} \right) + \frac{1}{2} d^{a+1} \gamma_2 \left( \frac{u}{d^{a-1}} \right) + \ldots, \quad (6)$$

an expansion of the type (5) except for the term with coefficient $d^{a-1}$. 

Gerd Grubb Copenhagen University

The transmission property
So $E_{a-1}(\bar{\Omega})$ contains $E_a(\bar{\Omega})$, differing just by having a nonvanishing term $d^{a-1}\varphi$ in the start. We conclude:

**Lemma 7.** $E_a(\bar{\Omega})$ is the subset of $E_{a-1}(\bar{\Omega})$ for which $\gamma_0\left(\frac{u}{d^{a-1}}\right)$ vanishes.

This extends by density to the $a$-transmission spaces. There are continuous maps

$$\gamma_{a,j} : u \mapsto \gamma_j\left(\frac{u}{d^a}\right)$$

from $H^{a(s)}_p(\bar{\Omega})$ to $B_{p}^{s-a-j-1/p}(\partial\Omega)$, when $s > a + j + 1/p$, and there holds:

**Lemma 8.** $H^{a(s)}_p(\bar{\Omega})$ is the subset of $H^{(a-1)(s)}_p(\bar{\Omega})$ for which $\gamma_0\left(\frac{u}{d^{a-1}}\right)$ vanishes.

The value $\gamma_0\left(\frac{u}{d^{a-1}}\right)$ is the generalized Dirichlet boundary value!

And we can show:

**Theorem 9.** The nonhomogeneous Dirichlet problem with $u$ sought in $H^{(a-1)(s)}_p(\bar{\Omega})$, 

\[
\begin{cases}
r^+ Pu = f \text{ on } \Omega, \\
\text{supp } u \subset \bar{\Omega}, \\
\gamma_0\left(\frac{u}{d^{a-1}}\right) = \varphi \text{ on } \partial\Omega,
\end{cases}
\]

is Fredholm solvable for $s > a - 1 + 1/p$ with data $f \in H^{s-2a}_p(\Omega)$, $\varphi \in B_{p}^{s-a+1-1/p}(\partial\Omega)$. 

Gerd Grubb Copenhagen University

The transmission property
Note here that when $u$ is such that $\gamma_0\left(\frac{u}{d^{a-1}}\right) = \varphi \neq 0$ at $x_0 \in \partial \Omega$, then $u(x)$ blows up like $d^{a-1}$ when $x \to x_0$. The solutions with nonzero Dirichlet data are “large” in this sense (also observed by Abatangelo ’15).

We can also study Neumann problems on $H^{(a-1)(s)}(\Omega)$:

$$
\begin{cases}
r^+ Pu = f \text{ on } \Omega, \\
\text{supp } u \subset \overline{\Omega}, \\
\gamma_1\left(\frac{u}{d^{a-1}}\right) = \psi \text{ on } \partial \Omega,
\end{cases}
$$

with $f \in \overline{H}_p^{s-2a}(\Omega)$, $\psi \in B_p^{s-a-1/p}(\partial \Omega)$.

There is Fredholm solvability at least when $P$ is principally like $(-\Delta)^a$. (G in A&PDE ’14)

One can check that when $u$ is in the smaller space $H^{a(s)}(\Omega)$ (i.e., when the Dirichlet value $\gamma_0\left(\frac{u}{d^{a-1}}\right)$ vanishes), then the Neumann value $\gamma_1\left(\frac{u}{d^{a-1}}\right)$ equals $\gamma_0\left(\frac{u}{d^{a}}\right)$.

The above boundary conditions are local. There exist other well-posed boundary conditions for $(-\Delta)^a$, of interest in probability theory, that are nonlocal.
Finally, just a few words on integration by parts. When $u$ is in the Dirichlet domain $H^{a(s)}(\Omega)$, then as noted, $\gamma_0\left(\frac{u}{d^a}\right)$ plays the role of a Neumann boundary value. We can show (G JDE’16):

**Theorem 10.** Let $P$ be a classical $\psi$do of order $2a \ (0 < a < 1)$ with even symbol, elliptic avoiding a ray. Then for $u, u' \in H^{a(s)}(\Omega), s > a + \frac{1}{2}$,

$$\int_{\Omega} Pu \partial_j \bar{u}' \, dx + \int_{\Omega} \partial_j u \overline{P^* u'} \, dx,$$

$$= \Gamma(a + 1)^2 \int_{\partial\Omega} s_0(x) \nu_j(x) \gamma_0\left(\frac{u}{d^a}\right) \gamma_0\left(\frac{\bar{u}'}{d^a}\right) \, d\sigma + \int_{\Omega} [P, \partial_j] u \bar{u}' \, dx,$$

where $[P, \partial_j]$ is the commutator $P\partial_j - \partial_j P$. Here $\nu_j$ is the $j$'th component of the normal vector $\nu$, and $s_0(x) = p_0(x, \nu(x))$.

It was shown first for $P = (-\Delta)^a$ by Ros-Oton and Serra ARMA ’14, then in a joint work with Valdinoci ’16 for translation-invariant selfadjoint positive homogeneous $P$, by integral operator methods. The new thing here is to allow variable coefficients, nonselfadjointness, lower-order terms, by $\psi$do methods — in particular finding the new term with $[P, \partial_j]$.

Such formulas are useful for nonexistence proofs in nonlinear problems.
There is a corollary with $\partial_j$ replaced by a radial derivative $x \cdot \nabla$, and of course other integral terms. As a corollary of this, we can show a Pohozaev-type formula for semilinear problems:

Consider the non-linear Dirichlet problem

$$r^+ Pu = f(u), \quad \text{supp } u \subset \overline{\Omega}, \quad (7)$$

where $f(s) \in C^{0,1}(\mathbb{R})$. Let $F(t) = \int_0^t f(s) \, ds$.

**Corollary 11.** Let $P$ be self-adjoint. Any bounded real solution $u$ of (7) satisfies the Pohozaev formula:

$$-2n \int_{\Omega} F(u) \, dx + n \int_{\Omega} f(u) \, u \, dx$$

$$= \Gamma(1 + a)^2 \int_{\partial \Omega} (x \cdot \nu) s_0 \gamma_0 (\frac{u}{d^a})^2 \, d\sigma + \int_{\Omega} [P, x \cdot \nabla] u \, u \, dx.$$

If $P$ is $x$-independent, the last term is replaced by $\int_{\Omega} \text{Op}(\xi \cdot \nabla \xi p) u \, u \, dx$.

**Example.** Applied to $P = (-\Delta + m^2)^a$ with $f(u) = \text{sign } u |u|^r$, this implies that on sharshaped domains, there are no nontrivial solutions in the critical and supercritical cases, where $r \geq \frac{n+2a}{n-2a}$. This follows from a sign analysis of the entering terms.
Details, if there is time: When \( p(\xi) = (|\xi|^2 + m^2)^a \),

\[
\xi \cdot \nabla p(\xi) = 2a|\xi|^2(|\xi|^2 + m^2)^{a-1} = 2a(|\xi|^2 + m^2)^a - 2am^2(|\xi|^2 + m^2)^{a-1},
\]

so \( \text{Op}(\xi \cdot \nabla p(\xi)) = 2aP - P_1 \), where \( P_1 = 2am^2(-\Delta + m^2)^{a-1} \) is a positive operator. The Pohozaev identity is then

\[
-2n \int_{\Omega} F(u) \, dx + (n - 2a) \int_{\Omega} f(u) \, u \, dx + \int_{\Omega} P_1 \, u \, u \, dx
\]

\[
= \Gamma(1 + a)^2 \int_{\partial \Omega} (x \cdot \nu) \, s_0 \gamma_0 (\frac{u}{d^a})^2 \, d\sigma.
\]

When \( f(u) = \text{sign } u \, |u|^r \) with an \( r > 1 \), \( F(u) = \frac{1}{r+1}|u|^{r+1} \), giving the formula

\[
\frac{-2n + (n - 2a)(r+1)}{r+1} \int_{\Omega} |u|^{r+1} \, dx + \int_{\Omega} P_1 \, u \, u \, dx = \Gamma(1 + a)^2 \int_{\partial \Omega} (x \cdot \nu) \, s_0 \gamma_0 (\frac{u}{d^a})^2 \, d\sigma.
\]

When \( r \geq \frac{n+2a}{n-2a} \), the left-hand side is positive unless \( u \equiv 0 \). For star-shaped \( \Omega \), the right-hand side is \( \leq 0 \). In such cases there are no nontrivial solutions.