Boutet de Monvel's contributions to the theory of Toeplitz operators

Victor Guillemin

MIT

The topic of this lecture:
Boutet de Monvel's contributions
to the theory of Toeplitz operators

In this lecture I'll focus for the most part on the abstract theory of Toeplitz operators, or alternatively, the theory of *Hermite* operators, a theory developed by Boutet and Boutet-Treves in the early- to mid-1970's.

I also want to spend some time discussing the *concrete* theory of Toeplitz operators, in particular, Boutet's paper with Sjöstrand, "Sur la singularité des noyaux de Bergman et Szego."

The operators figuring in this paper, by the way, are not Hermite operators per se, but a closely related class of operators: the *Fourier integral operators with complex phase* invented by Melin and Sjöstrand.

(But I'll be a little sloppy about differentiating between them since, in a lot of contexts they can be used interchangeably.)

Last but not least, I will talk a bit about my own collaboration with Boutet and our monograph in the Princeton University Press Series, *The Spectral Theory of Toeplitz Operators*.

To define Hermite operators I'll first have to define Hermite distributions, and I'll begin this talk by describing a result of Hormander which involves the first example I know of of a Hermite-like distribution. (In fact one can think of Hermite distributions as "superpositions" of this Hormander example.)

A quick review of some basic definitions.

Let μ be a distribution defined on an open subset X of $\mathbb{R}^n$ and let x_0 be a point of X.

Definition 1. $x_0 \notin$ sing. supp μ iff $\exists \rho \subset C_0^\infty(X)$ with $\rho(x_0) \neq 0$ and $\rho\mu \in C_0^\infty(X)$.

Alternatively: $x_0 \notin$ sing. supp μ iff $\exists \rho \subset C_0^\infty(X)$ with $\rho(x_0) \neq 0$ and $|\widehat{\rho\mu}(\xi)| \leq C_N(1+|\xi|)^{-N}$ for all N.

Hormander's refinement of this definition

Let $\xi_0 \in T^*_{x_0} - \{0\}$.

Definition 2. $(x_0, \xi_0) \notin WF(\mu)$ iff $\exists \rho \in C_0^\infty(X)$ with $\rho(x_0) \neq 0$ and $|\widehat{\rho\mu}(\xi)| \leq C_N(1+|\xi|)^{-N}$ for all N and for $\frac{\xi}{|\xi|} \doteq \frac{\xi_0}{|\xi_0|}$

Recall now the following result of Laurent Schwartz

Theorem. For any closed subset, Z, of X there exists a distribution, μ, with singular support Z.

Proof. Let $x_1, x_2, x_3, \ldots$ be a dense subset of Z and choose the a_i's in the sum below

$$\mu = \sum a_i \delta_{x_i} \quad a_i \in \mathbb{R}$$

so that the sum converges.

Hormander's generalization of this

Theorem. For *any* closed conic subset, Σ, of $T^*X - O$ there exists a distribution, μ, on X with $WF(\mu) = \Sigma$.

Proof. By an argument similar to (*) it suffices to show

Theorem. Given $\xi_0 \in T^*_{x_0}$ there exists a distribution, μ, on X with $WF(\mu) = \{(x_0, \lambda\xi_0), \lambda > 0\}$

Proof. Let $\mathbb{R}^n = \mathbb{R} \times \mathbb{R}^{n-1}$, $x = (t, y)$ and $\xi = (\tau, \eta)$, and consider the product

$$g(\xi) = f\left(\frac{\eta}{\sqrt{\tau}}\right)\rho(\tau)$$

with $f \in S(\mathbb{R}^{n-1}), \rho \in C^\infty(\mathbb{R})$ and

$$\rho(\tau) = \begin{cases} 0, & \tau < 1 \\ 1, & \tau > 2 \end{cases}$$

Theorem. The wave front set of the distribution

$$\mu = \left(\frac{1}{2\pi}\right)^n \int e^{ix\cdot\xi} g(\xi) d\xi$$

is the set, $x = 0, \eta = 0, \tau > 0$

Proof. By the Fourier inversion formula $g = \widehat{\mu}(\xi)$ and g is rapidly decreasing along all rays for which $\eta \neq 0$ and is zero for $\tau < 0$. Hence $WF(\mu)$ is contained in the set $\eta = 0, \tau > 0$.

Moreover for $D = \frac{1}{\sqrt{-1}} \left(\frac{\partial}{\partial \xi_1}, \ldots, \frac{\partial}{\partial \xi_n} \right)$

(I) $|D^\alpha g(\xi)| \leq C_\alpha (1 + |\xi|)^{-\frac{|\alpha|}{2}}$

and

(II) $x^\alpha \mu = \left(\frac{1}{2\pi}\right)^n \int D^\alpha g e^{ix\cdot\xi} d\xi$

and by (I) and (II) μ is C^∞ for $x \neq 0$.

I'll now generalize this result a bit: Let Σ_0 be a k dimensional subspace of $T_0^*\mathbb{R}^n$ and choose coordinates $x = (t, y)$ on $\mathbb{R}^n$ and dual coordinates $\xi = (\tau, \eta)$ on $(\mathbb{R}^n)^+$ so that Σ_0 is the set $x = 0$ and $\xi = (\tau, 0)$

Let $f_m(\xi) = f_m(\tau, \eta)$ be a C^∞ function which is homogeneous of degree m in τ for $|\tau| \gg 0$, is zero for τ small and is rapidly decreasing in η and let $f(\xi)$ be a C^∞ function which admits an asymptotic expansion

$$f(\tau, n) = \sum_{i=0}^{-\infty} f_{m_i}(\tau, \eta)$$

with $m_0 = m - \frac{n}{2}$ and $m_i \to -\infty$

Replacing the μ in the discussion above by

$$\mu = \left(\frac{1}{2\pi}\right)^n \int e^{ix\cdot\xi} f\left(\tau, \frac{\eta}{\sqrt{|\tau|}}\right) d\xi \tag{1}$$

one gets essentially the same proof as above:

Theorem $WF(\mu) \subseteq \Sigma_0$

Following Boutet we'll denote the space of these distributions by $I^m(\mathbb{R}^n, \Sigma_0)$

The elements of this space are the prototypical examples of *Hermite* distributions.

The definition of “Hermite” in general: Let X be an n-dimensional manifold and let Σ be a conic isotropic submanifold of $T^*X - 0$

Definition. A distribution, μ, on X is in $I^m(X, \Sigma)$ if, for every (x, ξ) in Σ there exists a $(0, \xi_0)$ in Σ_0, a conic neighborhood, $\mathcal{U}$, of (x, ξ) in $T^*X - 0$, a conic neighborhood, $\mathcal{U}_0$, of X_0, ξ_0 in $T^*\mathbb{R}^n - 0$ and a homogeneous canonical transformation $\phi : \mathcal{U} \to \mathcal{U}_0$ mapping $\mathcal{U} \cap \Sigma$ onto $\mathcal{U}_0 \cap \Sigma_0$ such that, for every zeroth order F.I.O. with microsupport on graph ϕ, $F_\mu \in I^m(\mathbb{R}^n, \Sigma_0)$

i.e. μ is in $I^m(X, \Sigma)$ if, at every point $(x, \xi) \in \Sigma$ it is micolocally isomorphic to a distribution, $\mu_0 \in I^m(\mathbb{R}^n, \Sigma_0)$

Hermite operators

Let X be an n-dimensional manifold and $\Sigma \subseteq T^*X - 0$ a closed conic symplectic submanifold of T^*X. We will identify Σ with the isotropic submanifold

$$\Sigma = \{(x, \xi, x, -\xi), (x, \xi) \in \Sigma\}$$

of $T^*(X \times X) - 0$ and define Hermite operators with microsupport on Σ to be operators of the form

$$T_K : C^\infty(X) \to C^\infty(X)$$

where K is a distributional Kernel in $I(X \times X, \Sigma)$

How do these operators come up in practice? To answer this let me turn first to the paper of Boutet de Monvel: “Hypoelliptic operators with double characteristics and related pseudo-differential operators” *CPAM* (1974)

For lack of time I won't attempt tp define "hypoelliptic operators, P, with double characteristics" except to say that they occur in two flavors: "involutive" if the characteristic variety of P is an involutive submanifold of $T^*X - O$ and "symplectic" if it is symplectic. I'll also mention for future use that an operator of the second kind is the "$\bar{\partial}_b$ Laplacian"

What is the role of Hermite operators in this subject?

In his introduction to this paper Boutet parenthetically remarks that for hypoelliptic operators of the second kind “Hermite operators provide a precise description of their Kernels and co Kernels mod C^∞”. To see how this works in the example above I’ll turn next to:

L. Boutet de Monvel and J. Sjöstrand, *Sur la singularité des noyaux de Berman et de Szego*

The introduction to this paper: Some basic definitions

Soit $\Omega \subseteq \mathbb{C}^n$ un ouvert borne en $\mathbb{C}^n$, de frontiere $C^\infty, X = \partial\Omega$, strictement pseudoconvex. Ω est donc defini par une inequation $\rho < 0$ ou ρ est une fonction reelle C^∞ sur $\mathbb{C}^n$ et $d\rho \neq 0$ si $\rho = 0$. (Donc X est le varieté $\rho = 0$)

Nous rappelons:

Si $f \in C^\infty(\bar{\Omega}) \cap O(\Omega)$, la trace $f|X$ verify les equation de Cauchy-Riemann induit, $\bar{\partial}_b(f|X) = 0$.

Maintenent soit:

$$B : L^2(\bar{\Omega}) \to L^2(\bar{\Omega}) \cap O(\mathcal{U})$$

et

$$S : L^2(X) \to L^2(X) \cap \text{Ker } \bar{\partial}_b$$

les projections orthogonales des espaces au gauche sur les sous-espaces au droite (les projecteurs de *Bergman* et *Szego*).

Avec ses notations Boutet et Sjöstrand demontrent

Theoreme S est un operateur integrale de Fourier a phase complexe dans le seus de Melin-Sjöstrand (ou, alternativement, un operateur de type Hermite associé a la variete symplectique conique, $\left\{(x,\xi), x \in X, \xi = i\lambda\bar{\partial}\rho_x|X\right\}$.)

Puis ils remarquent:

> *“On passe de la au noyau de Bergman en exploitant le fait qu’une fonction holomorphe est harmonique, donc s’exprime en fonction de sa trace sur le bord au moyen du noyau de Poisson.”*

Using microlocal techniques they are able to obtain a very precise description of the asymptotic properties of S and then, exploiting the remark above, obtain analogous asymptotic properties for B.

Some consequences. Let ρ as above be the defining function for $bd(\Omega) = X$ and $\psi(x, y) \subset C^\infty(\mathbb{C}^n \times \mathbb{C}^n)$ a function with the properties

1. $\psi(x, x) = \rho(x)$
2. $\bar{\partial}_x \psi$ and $\partial_y \psi$ vanish to infinite order at $x = y \in X$
3. $\psi(x, y) = \bar{\psi}(y, x)$

Theoreme Il existe des fonctions, F, G sur $X \times X$ (resp. F', G' sur $\bar{\Omega} \times \bar{\Omega}$) telle que [1]

$$S = F(-i\psi)^{-n} + G(-i\psi)$$

et

$$B = F'(-i\psi)^{-n-1} + G' \text{ Log } (-i\psi)$$

[1] Sur $X \times X$ on note un peu abusivement que $(-i\psi)^{-n}$ est la distribution limite pour $\epsilon \to 0+$ de $(-\psi + \epsilon)^{-n}$

One well known corollary of this result is the theorem of Fefferman

$$B(z, \bar{z}) = F'(z, \bar{z})\rho^{-n-1} + G'(z, \bar{z}) \log \rho$$

and in fact provides a microlocal proof of this result.

A quote (from Charles Epstein's review of the paper above in the Boutet memorial volume)

> *"Beginning with the papers of Catlin and Zelditch in the 1990's the FIO constructions of these kernel functions has proved extremely useful. This paper was cited only 5 times between its publication and 1997 and 110 times between 1998 and 2014."*

I'll now come back to the theory of abstract Toeplitz operators and say a few words about my monograph with Boutet. Let X be a compact manifold and Σ a closed conic symplectic submanifold of $T^*X - O$. Then one can construct abstracrt analogues of the Szego projector, S, namely Hermite operators, Π, with microsupport on Σ satisfying

$$\Pi = \Pi^t = \Pi^2$$

In addition on can show that the set of operators

$$\{T = \Pi P \Pi,\ P \in \Psi(X)\}$$

is an *algebra* having a lot of properties of the algebra of pseudodifferential operators, $\Psi(X)$, itself. In particular to each operator, $T = \Pi P \Pi$ one can attach a symbol

$$\sigma(T) = \sigma(P)|Z$$

Moreover these symbols have a lot of the same properties as symbols of pseudodifferential operators, e.g.

$$\sigma(T_1)\sigma(T_2) = \sigma(T_1 T_2)$$

In [BG] we show also that a lot of basic theorems in the *spectral theory* of pseudodifferential operators on compact manifolds have Toeplitz analogues. I'll content myself here by describing three such theorems.

Theorem 1. (Weyl's theorem) Let T be a self adjoint first order Toeplitz operator with $\sigma(T) > 0$ and let $\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ be its spectrum. Then if $N(\lambda)$ is the number of λ_i's less than λ

$$N(\lambda) = \frac{\mathrm{vol}(\Sigma_1)}{(2\pi)^n}\lambda^n + O(\lambda^{n-1})$$

Σ_1 being the subset, $\sigma(T) = 1$ of the cone Σ.

Theorem 2. (Helton's theorem) Let Ξ be the Hamiltonian vector field on the symplectic cone Σ associated with $\sigma(T)$ and let Λ be the set of cluster points of the set $\{\lambda_i - \lambda_j\}$. Then $\Lambda \neq \mathbb{R}$ only if the trajectory of Ξ through every point of Σ is periodic.

Theorem 3. (The wave trace theorem)

The distribution

$$\Sigma e^{\sqrt{-1}\lambda_i t}$$

has singular support contained in the set of *periods*, T_γ, of the period trajectories of $\exp \Xi$. Moreover if these periodic trajectories are simple

$$\Sigma e^{\sqrt{-1}\lambda_i t} = \Sigma \alpha_\gamma (t - T_\gamma + \sqrt{-1}\, 0+)^{-1} + \dots$$

where "…" is locally summable.

Moreover, there are simple formulas for the a_γ's (which I won't have time to describe here).

The results above are, as I have mentioned, contained in my monograph with Louis, *The Spectral Theory of Toeplitz Operators*.

There is also an appendix to this monograph, written by Louis, on "quantized contact structures" and I'd like to conclude by describing briefly some of the items discussed in this appendix.

First a word about *quantized contact structures:* Let X be a compact $2n-1$ dimensional manifold and $\alpha \in \Omega^1(X)$ a contact form. Then the set

$$\Sigma = \left\{(x, \lambda\alpha_x), x \in X, \lambda \in \mathbb{R}^+\right\}$$

is a conic symplectic submanifold of $T^*X - 0$ and hence, by the techniques described earlier in this lecture, can be quantized.

In other words on can associate to Σ a generalized Szego projector $\Pi : L^2(X) \to H^2(X)$ and an algebra of Toeplitz operators on $H^2(X)$

$$\Pi P \Pi,\ \ P \in \Psi(X)$$

Example: Let $\Omega \subseteq \mathbb{C}^n$ be a strictly pseudoconvex domain with C^∞ boundary, X.

Then if ρ is as above a defining function for X,

$$\alpha = \sqrt{-1}\bar{\partial}\rho | X$$

is a contact form and we can take for the quantization of (X, α) the Szego projector, S, and the algebra of Toeplitz operators, SPS.

In what I've discussed so far the Π figuring in the quantization of a contact manifold (X, α) can be assumed to be either a Hermite operator or a Fourier integral operator with complex phase, but for what I'll be describing next I'll have to assume the latter is the case.

Making this assumption Boutet shows that this quantization has some other striking similarities with the quantization of the boundary, X, of a pseudoconvex domain other than those I've already discussed. Namely for a pseudoconvex domain one has a $\bar{\partial}_b$ complex

$$C^\infty(X) \xrightarrow{\bar{\partial}_b} C^\infty(X, \Lambda_b^{0,1}) \xrightarrow{\bar{\partial}_b} \cdots \tag{2}$$

and in his article “On the index of Toeplitz operator of several complex variables”, *Inventiones Math.* (1979) he uses this complex to prove a Toeplitz version of the Atiyah-Singer theorem.

One of the main results in his appendix to our monograph:

An analogue of this complex exists for *arbitrary* quantized contact structures.

To end on a personal note:

The first draft of this monograph was written in the spring of 1978 when both of us were participating in a ”Year in Microlocal Analysis” at the IAS and the following summer he invited me to spend a month in Paris with him and during this time we got the monograph into publishable form and Louis put the finishing on the appendix.

This summer was the first of many summer visits to Paris in which I had the chance to continue to be in contact with Louis; and although they didn't result in a "Theory of Toeplitz operators, part 2" I acquired from these visits and my contact with him many new and beautiful insights.

I should also mention that it was during this period 1978–1979, that I got to know well two of my current collaborators, Johannes and Anne, with whom I've been collaborating on the Boutet memorial volume and which, thanks largely to their efforts, is soon to see the light of day.