Simplifications of the Keiper/Li approach to the Riemann Hypothesis (RH)

André Voros
CEA-Institut de Physique Théorique de Saclay

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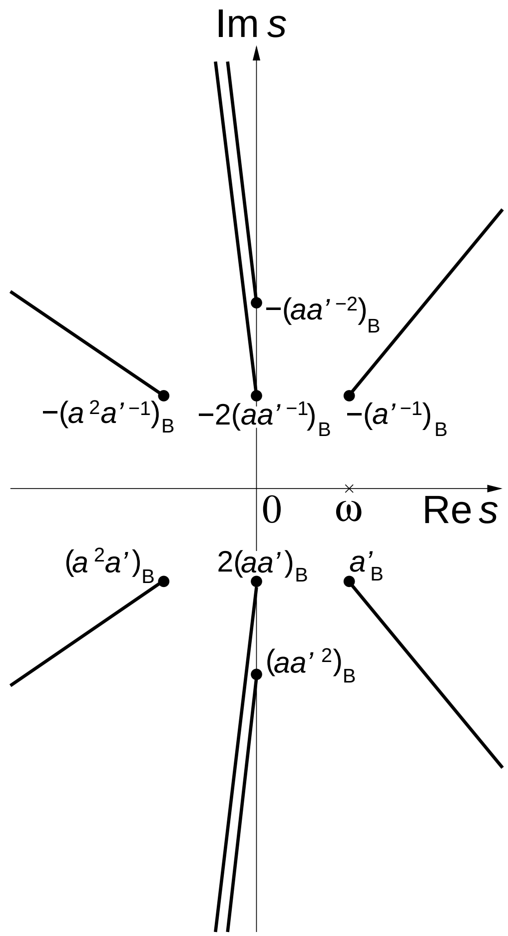
IPhT15/106, HAL cea-01166324; IPhT16/011, arXiv:1602:03292 [math.NT]

NOT ABOUT proving RH / relevance for number theory of RH
ONLY ABOUT criteria (rewordings, equivalent forms, tests) for RH

Exact asymptotics

(Leray, Sato–Kawai–Kashiwara, Dingle, Balian–Bloch, Sibuya, Zinn-Justin)

Example : $-\frac{\mathrm{d}^2}{\mathrm{d}q^2} + V(q)$ for $V(q) = q^4$ (1D pure-quartic oscillator)



1D exact asymptotics for $-\psi_\lambda''(q) + (V(q) + \lambda)\,\psi_\lambda(q) = 0$

(polynomial potential V)

Wronskian identity:

$$\psi_\lambda(q)\Psi_\lambda'(q) - \psi_\lambda'(q)\Psi_\lambda(q) \equiv 2\mathrm{i}.$$

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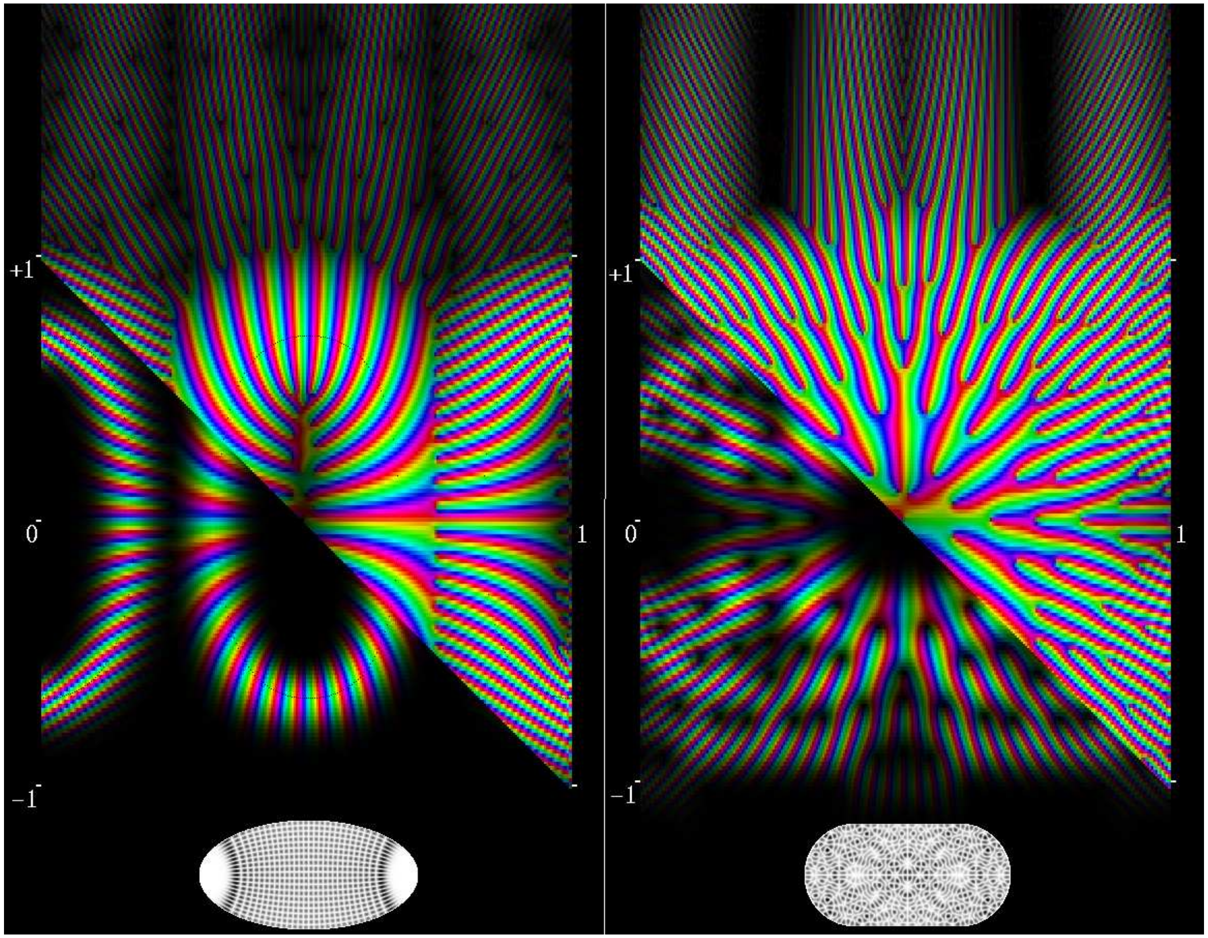
$$\Downarrow$$

Zeros (exact quantization conditions or **Bethe Ansatz**)

The multidimensional puzzle



The Riemann zeta function $\zeta(x)$

$$\zeta(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} k^{-x} \qquad (\mathrm{Re}\ x > 1)$$

$$\equiv \prod_{\{p\}} (1-p^{-x})^{-1} \quad \text{(Euler product: i.e., } \log\zeta \text{ encodes the primes)}$$

$$\equiv \frac{1}{\Gamma(x)} \int_0^{\infty} \frac{1}{\mathrm{e}^u - 1} u^{x-1}\, \mathrm{d}u \qquad \text{(Mellin transformation)}$$

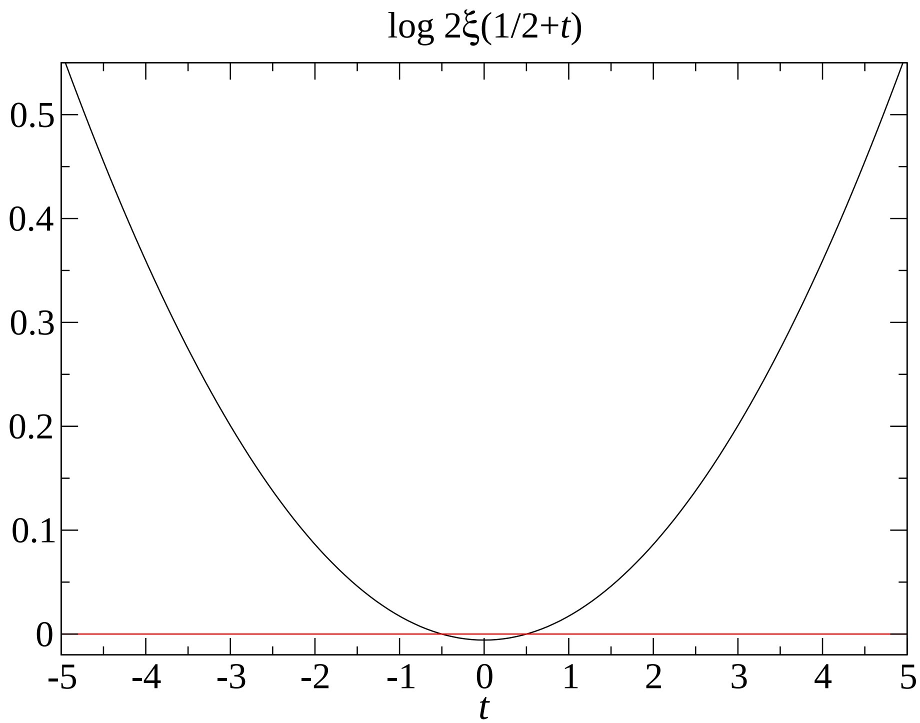
which extends ζ to $\mathbb{C}$ with the only singularity $\zeta(x) = \dfrac{1}{x-1} + \cdots$, and

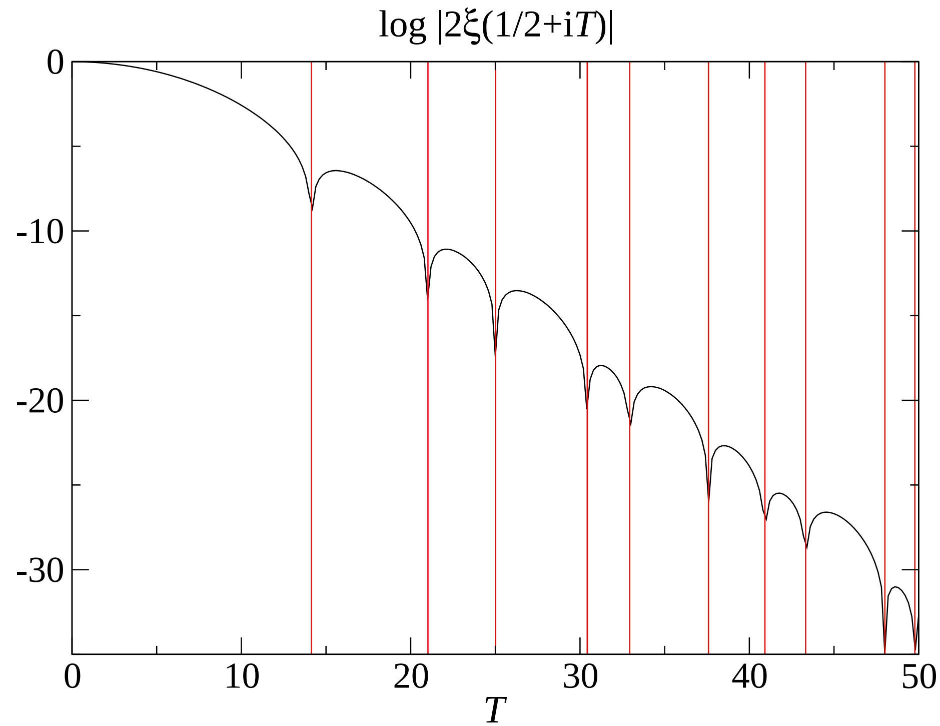
$$\zeta(-n) = (-1)^n B_{n+1}/(n+1) \qquad \text{for } n = 0, 1, 2, \ldots$$

Riemann's Functional Equation: ($\Leftarrow$ Poisson's summation formula)

$\boxed{\xi(x) \equiv \xi(1-x)}$ for $2\xi(x) \stackrel{\text{def}}{=} x(x-1)\pi^{-x/2}\Gamma(x/2)\,\zeta(x)$ (completed zeta function).

ξ is an entire function, with these **symmetries** about $x = \frac{1}{2}$:
central symmetry, reality, $\Rightarrow$ symmetry / the *critical line* $\{\mathrm{Re}\ x = \frac{1}{2}\}$.



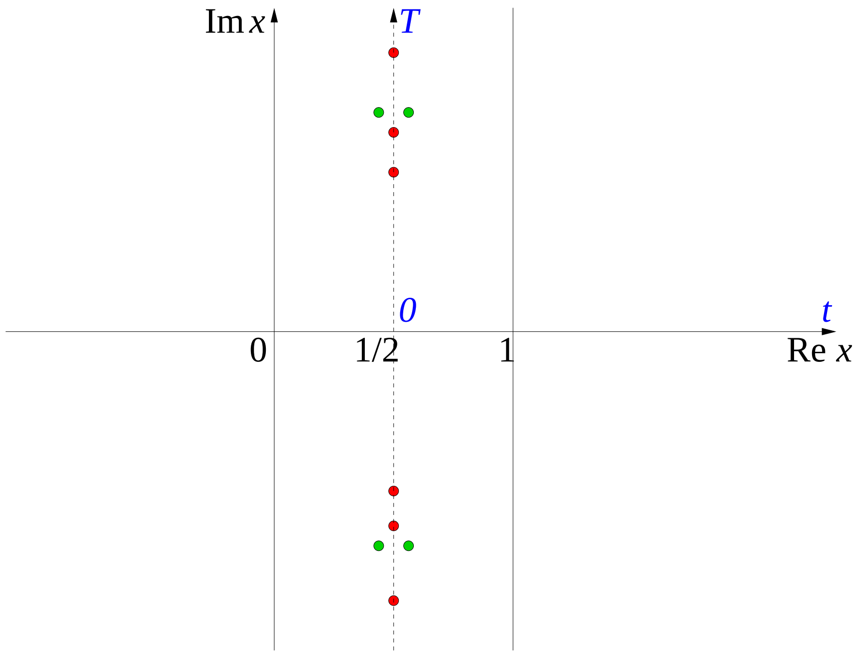


Riemann zeros $=$ zeros of $\xi(x)$ (denoted $\{\rho\}$), the lowest ones being

$\rho = \frac{1}{2} \pm \mathrm{i}T, \quad T \approx 14.1347251,\ 21.0220396,\ 25.0108576,\ 30.4248761,\ 32.9350616, \ldots$

The Riemann zeros - known facts

- Countably many zeros, all in the open *critical strip* $\{0 < \mathrm{Re}\ x < 1\}$.



- *Riemann–von Mangoldt theorem* for the zeros' counting function $N(T)$:

$$N(T) = \frac{T}{2\pi}\left(\log \frac{T}{2\pi} - 1\right) + \delta N(T), \quad \delta N(T) = O(\log T) \text{ as } T \to +\infty.$$

- *Hadamard product formula:* $\quad 2\xi(x) \equiv \prod_\rho (1 - x/\rho), \quad \prod_\rho \stackrel{\text{def}}{=} \lim_{T\to\infty} \prod_{|\,\mathrm{Im}\ \rho| \le T}.$

The Riemann Hypothesis (RH) (1859)

All the zeros ρ *of* $\xi(x)$ *lie on the axis* $\{\mathrm{Re}\, x = \frac{1}{2}\}$ (the **critical line**)

Numerically verified up to the 10^{13}-th zero or the height $T_0 \approx 2.4 \cdot 10^{12}$ (Gourdon 2004).

Significance for number theory: the Prime Number Theorem for the primes' counting function $\pi(k)$ states

$$\pi(k) \sim \mathrm{Li}(k) \stackrel{\text{def}}{=} \int_2^k \frac{\mathrm{d}v}{\log v} \quad \left(\approx \frac{k}{\log k}\right) \qquad \text{for } k \to \infty.$$

Then, **RH optimizes the error term** $\varepsilon(k) = \pi(k) - \mathrm{Li}(k)$:

$$\beta_0 \stackrel{\text{def}}{=} \sup_\rho \{\mathrm{Re}\, \rho\} \equiv \inf\{\beta \in \mathbb{R} \mid \varepsilon(k) = O(k^\beta)\},$$

and RH ($\iff \beta_0 = \frac{1}{2}$) more precisely amounts to $\varepsilon(k) = O(k^{1/2} \log k)$: the least possible fluctuation for the primes' distribution function $\pi(k)$ around its mean $\mathrm{Li}(k)$.

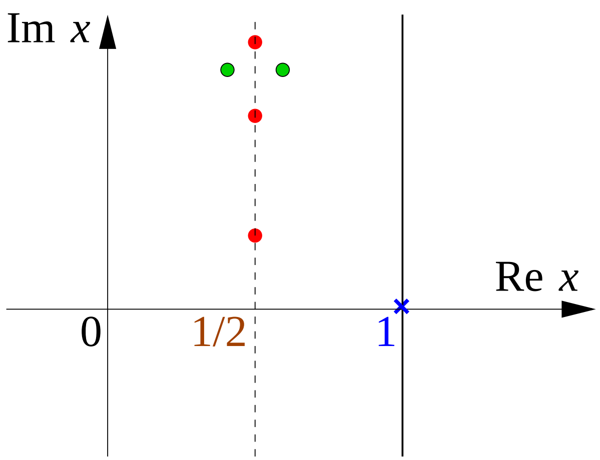
But currently: RH true or not? $\beta_0 < 1$ or $\beta_0 = 1$?

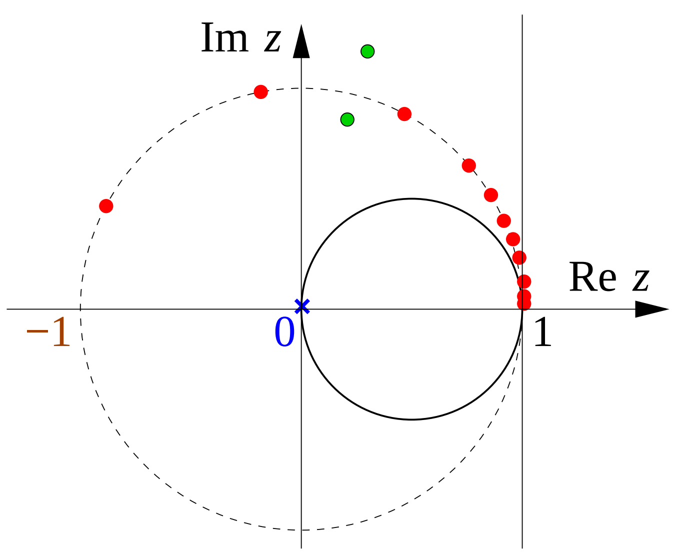
The Keiper/Li sequence as a testing tool for RH false

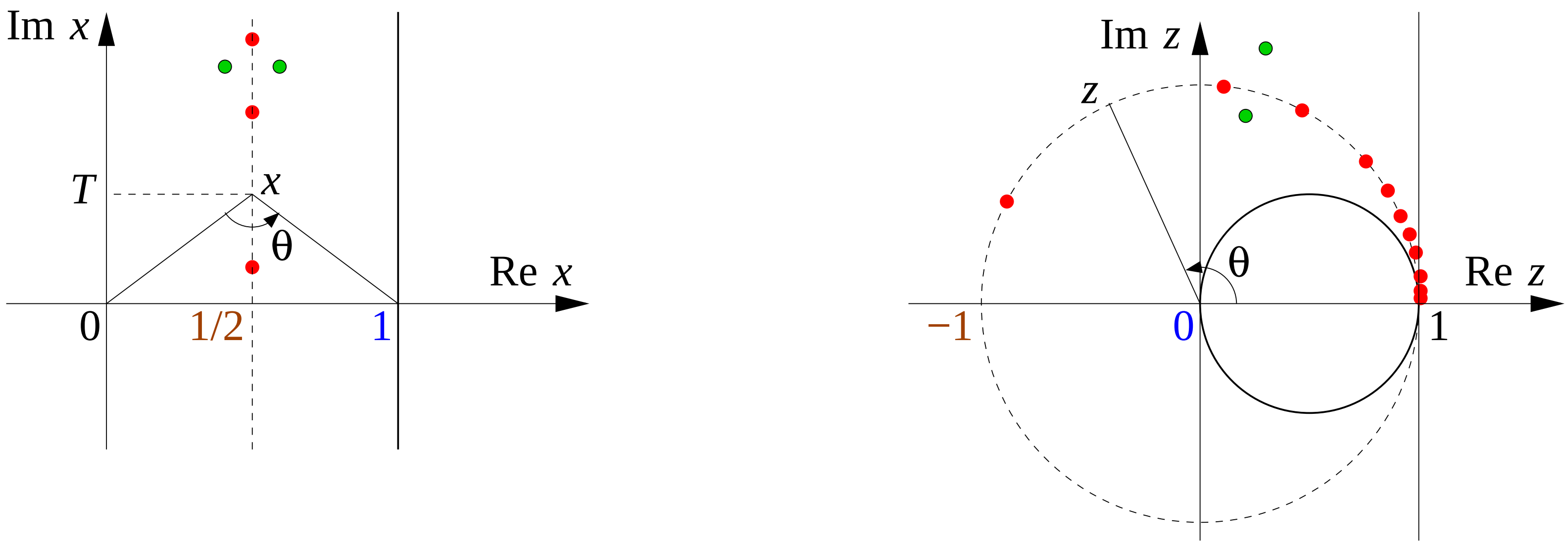
Defined by the (real) generating function (Keiper 1992, X.-J. Li 1997)

$$\sum_{n=1}^{\infty} \lambda_n \, z^{n-1} \equiv F(z) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z} \log \xi\left(x = \frac{1}{1-z}\right) \iff \lambda_n = \sum_{\rho} \left[1-(1-1/\rho)^n\right].$$

$$\lambda_1 = 1-\tfrac{1}{2}\log 4\pi+\tfrac{1}{2}\gamma \approx 0.0230957, \quad \lambda_2 \approx 0.0923457, \quad \lambda_3 \approx 0.207639, \ldots$$







$$x = \frac{1}{1-z} \qquad \Longleftrightarrow \qquad z = 1 - 1/x$$

$$x = \tfrac{1}{2} + \mathrm{i}T \qquad \Longleftrightarrow \qquad z = \mathrm{e}^{\mathrm{i}\theta}, \qquad T \equiv \tfrac{1}{2}\cot\,\tfrac{1}{2}\theta.$$

$$\rho = \tfrac{1}{2} \pm \mathrm{i}T_\rho \qquad \Longleftrightarrow \qquad z_\rho = \mathrm{e}^{\pm \mathrm{i}\theta_\rho}; \qquad \mathrm{Re}\ \rho = \tfrac{1}{2} \Longleftrightarrow \theta_\rho \text{ real.}$$

$$\lambda_n = \sum_\rho \left[1 - (1 - 1/\rho)^n\right] \equiv \sum_\rho (1 - z_\rho^n) \equiv \sum_\rho (1 - \cos n\theta_\rho).$$

If RH is true...

$$\lambda_n = \sum_\rho (1 - \cos n\theta_\rho) \qquad \text{with all } \theta_\rho \textit{ real}$$

implies, for the λ_n,

- positivity: $\boxed{\lambda_n > 0 \quad \text{for all } n}$ (Keiper 1992)
- large-order behavior through the integral formula (Oesterlé 2000)

$$\lambda_n = 2\int_0^\infty (1 - \cos n\theta)\, \mathrm{d}N(T) \quad \Longrightarrow \quad \frac{\lambda_n}{n} = 2\int_0^\pi \sin n\theta\, N(\tfrac{1}{2}\cot\tfrac{1}{2}\theta)\, \mathrm{d}\theta$$

and using $N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \delta N(T)$, $\quad \delta N(T) = O(\log T)_{T\to+\infty}$:

$$\boxed{\frac{\lambda_n}{n} \sim \tfrac{1}{2}\log n + \tfrac{1}{2}(\gamma - \log 2\pi - 1)}$$

Li's criterion (X.-J. Li 1997)

$$\textbf{RH true} \iff \lambda_n > 0 \text{ for all } n$$

Practical aspects:

- RH verified up to a height $T_0 \implies \lambda_n > 0$ as long as $n < T_0^2$ (Oesterlé)

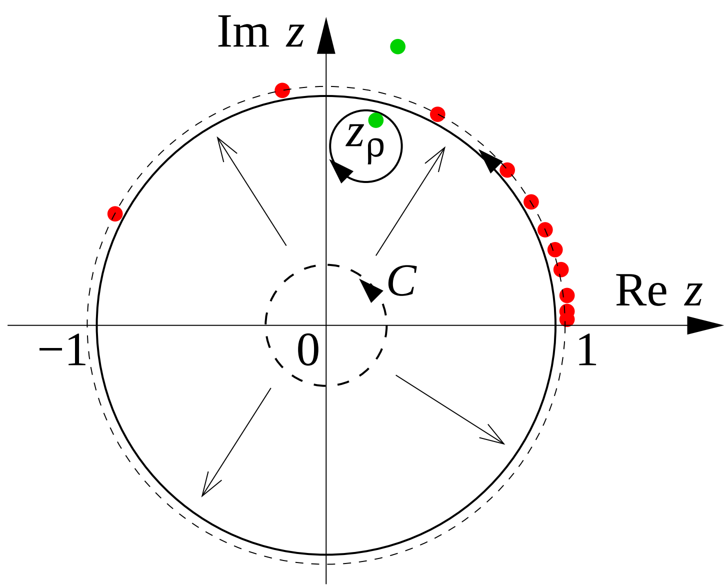
- a Riemann zero violating RH, $\rho = \frac{1}{2} + t \pm \mathrm{i}T$ with $0 < t$ $(< \frac{1}{2})$ (hence $T > T_0$), will practically be undetectable through λ_n unless

$$n \gtrsim T^2/t > 2T^2 \qquad (\gtrsim 10^{25} \text{ currently} : T_0 \approx 2.4 \cdot 10^{12}).$$

Asymptotic sensitivity to RH

Large-order behavior of λ_n : using Darboux's method for $F(z) \stackrel{\mathrm{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z} \log \xi(\frac{1}{1-z})$, a meromorphic function having simple poles of residue 1 at all images z_ρ of Riemann zeros,

$$\lambda_n = \frac{1}{2\pi \mathrm{i}} \oint_C z^{-n} F(z)\,\mathrm{d}z = - \sum_{\{|z_\rho|<1\}} z_\rho^{-n} \quad + o(r^{-n})_{n\to\infty} \text{ for all } r<1.$$



Asymptotic alternative (AV 2004)

$$\lambda_n \sim \begin{cases} -\sum_{\{|z_\rho|<1\}} z_\rho^{-n} \quad + o(r^{-n})_{n\to\infty} \text{ for all } r<1 & \textbf{if RH false} \\ \textit{(exponentially growing oscillations of both signs)} & \\ \\ \frac{1}{2}n\log n + \frac{1}{2}n(\gamma - \log 2\pi - 1) \quad + O(n^{1/2}\log n) & \textbf{if RH true} \\ \textit{(tempered growth to } +\infty\textit{)} \qquad \text{(Lagarias 2007)} & \end{cases}$$

Computing the λ_n

$$\lambda_n = 1 - \tfrac{1}{2}(\log 4\pi + \gamma)n + \sum_{j=2}^{n} (-1)^j \binom{n}{j} (1 - 2^{-j})\, \zeta(j) - \sum_{j=1}^{n} \binom{n}{j} \eta_{j-1}$$

(Bombieri–Lagarias 1999)

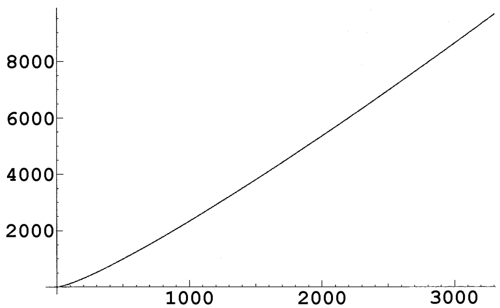
$$\log[(x-1)\,\zeta(x)] \quad \equiv \quad -\sum_{n=0}^{\infty} \frac{\eta_n}{n+1} (x-1)^{n+1}$$

vs $$(x-1)\,\zeta(x) \quad \equiv \quad \sum_{n=0}^{\infty} \gamma_n (x-1)^{n+1}$$ ((Stieltjes constants: $\gamma_0 \equiv \gamma$)

In terms of the *von Mangoldt function* $\Lambda(k) \stackrel{\text{def}}{=} \begin{cases} \log p & \text{if } k = p^r,\ p \text{ prime} \\ 0 & \text{otherwise :} \end{cases}$

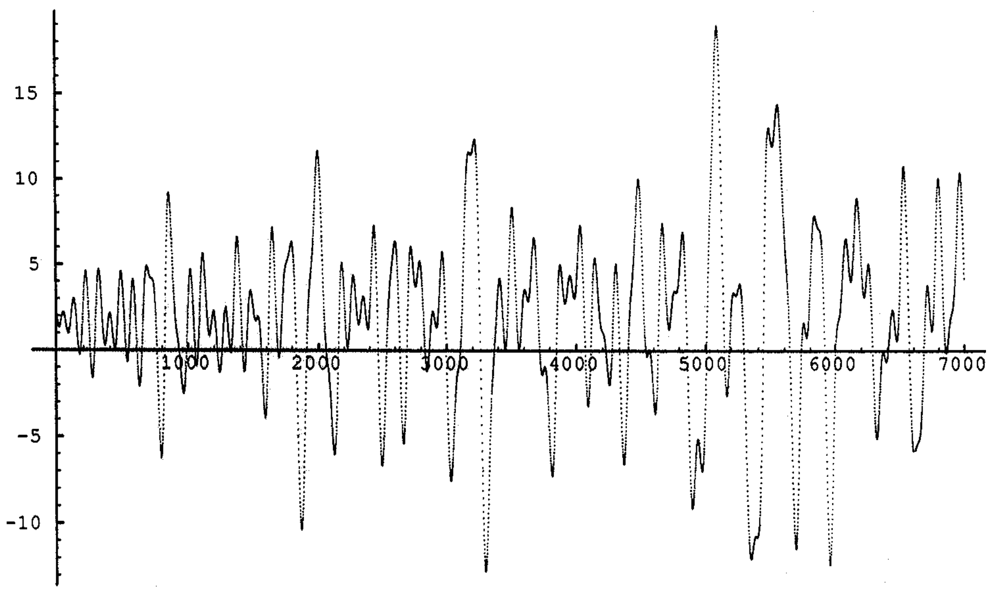
$$\eta_n = \frac{(-1)^n}{n!} \lim_{K\to\infty} \left\{ \sum_{k=2}^{K} \Lambda(k) \frac{(\log k)^n}{k} - \frac{(\log K)^{n+1}}{n+1} \right\} \qquad (\eta_0 \equiv -\gamma).$$

Numerical computations (also: Coffey 2005)



$$\eta_0 = -\gamma_0$$
$$\eta_1 = +\gamma_0^2 - 2\gamma_1$$
$$\eta_2 = -\gamma_0^3 + 3\gamma_0\gamma_1 - 3\gamma_2$$
$$\eta_3 = +\gamma_0^4 - 4\gamma_0^2\gamma_1 + 2\gamma_1^2 + 4\gamma_0\gamma_2 - 4\gamma_3$$
$$\eta_4 = -\gamma_0^5 + 5\gamma_0^3\gamma_1 - 5\gamma_0\gamma_1^2 - 5\gamma_0^2\gamma_2 + 5\gamma_1\gamma_2 + 5\gamma_0\gamma_3 - 5\gamma_4,$$
$$\ldots$$

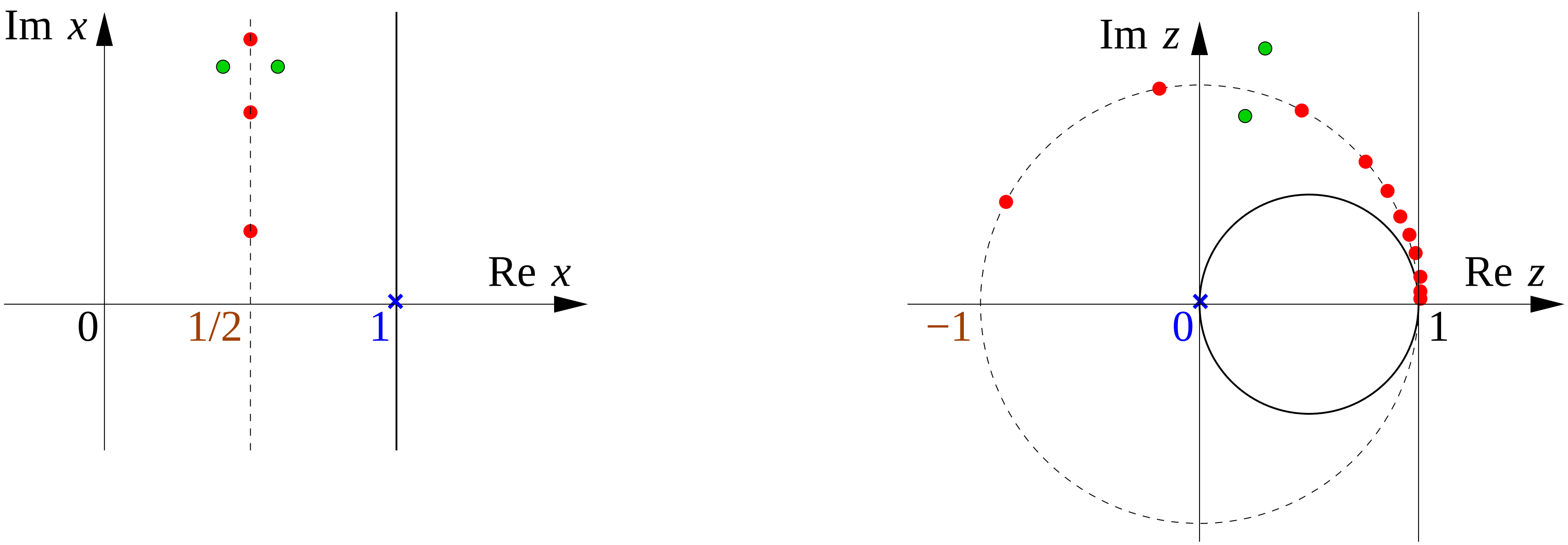
Maślanka 2004



Keiper 1992

Deformations of the λ_n : allowed?

$$\lambda_n = \frac{1}{(n-1)!}\left[x^{n-1}\log 2\xi(x)\right]^{(n)}_{x=1} \iff \sum_{n=1}^{\infty} \lambda_n\, z^{n-1} \equiv \frac{\mathrm{d}}{\mathrm{d}z}\log 2\xi\Big(\frac{1}{1-z}\Big)$$



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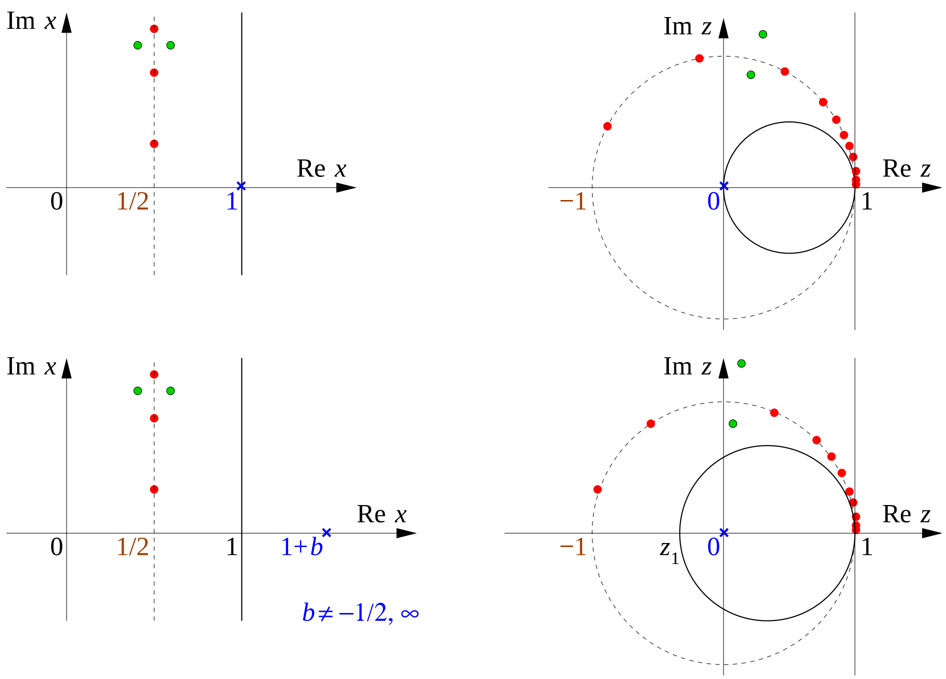
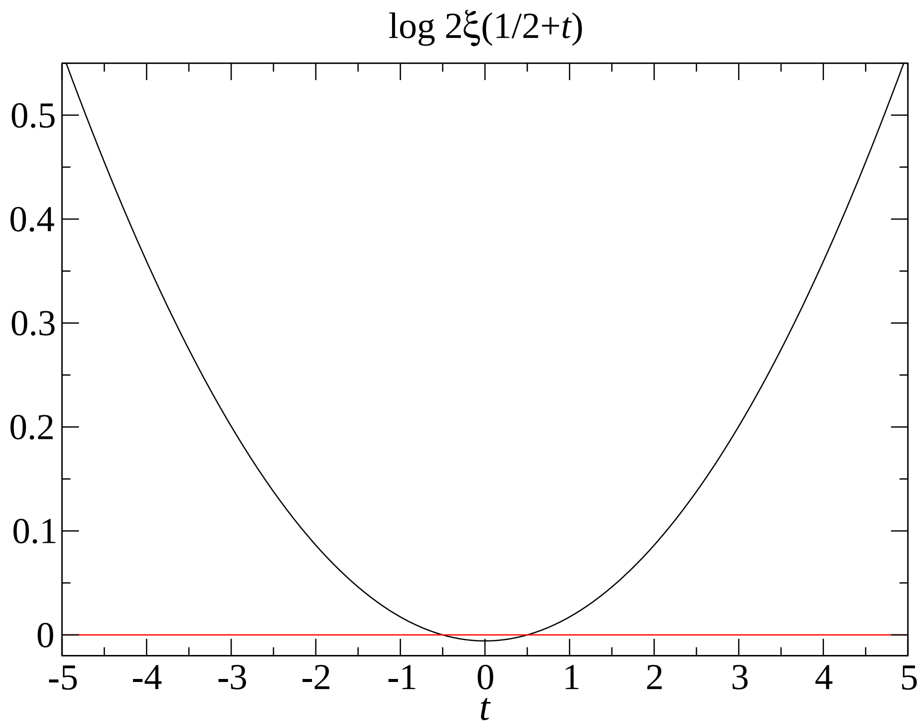


$b \neq -\frac{1}{2}$ (Sekatskii 2014)

$$\lambda_n^{(b)} = \frac{2b+1}{(n-1)!}\left[(x+b)^{n-1}\log 2\xi(x)\right]^{(n)}_{x=1+b} \iff \sum_{n=1}^{\infty}\lambda_n^{(b)}\, z^{n-1} \equiv \frac{\mathrm{d}}{\mathrm{d}z}\log 2\xi\Big(\frac{bz+1+b}{1-z}\Big)$$

Numerical observations



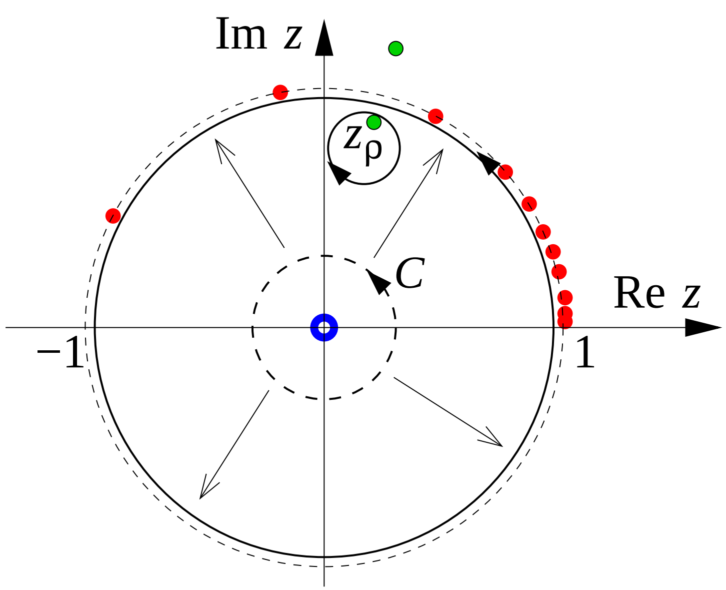
$$\begin{aligned}
\tfrac{1}{2}(\log 2\xi)''(\tfrac{1}{2}) &= 0.0231050\ldots \\
-4\log 2\xi(\tfrac{1}{2}) &= 0.0231003\ldots \\
(\log 2\xi)'(1) \equiv \lambda_1 &= 0.0230957089661210\ldots
\end{aligned}$$

We propose broader deformations

$$\lambda_n = \frac{1}{(n-1)!}\left[x^{n-1}\log 2\xi(x)\right]^{(n)}\Big|_{x=1} \iff \sum_{n=1}^{\infty}\lambda_n\, z^{n-1} \equiv \overbrace{\frac{\mathrm{d}}{\mathrm{d}z}\log 2\xi(}^{F}\,\overbrace{x(z)}^{\frac{1}{1-z}}\,),$$

$$\lambda_n = \frac{1}{2\pi \mathrm{i}}\oint_C \frac{\mathrm{d}z}{z^n}\,F(x(z)), \qquad \iff$$

Denominator: z^n



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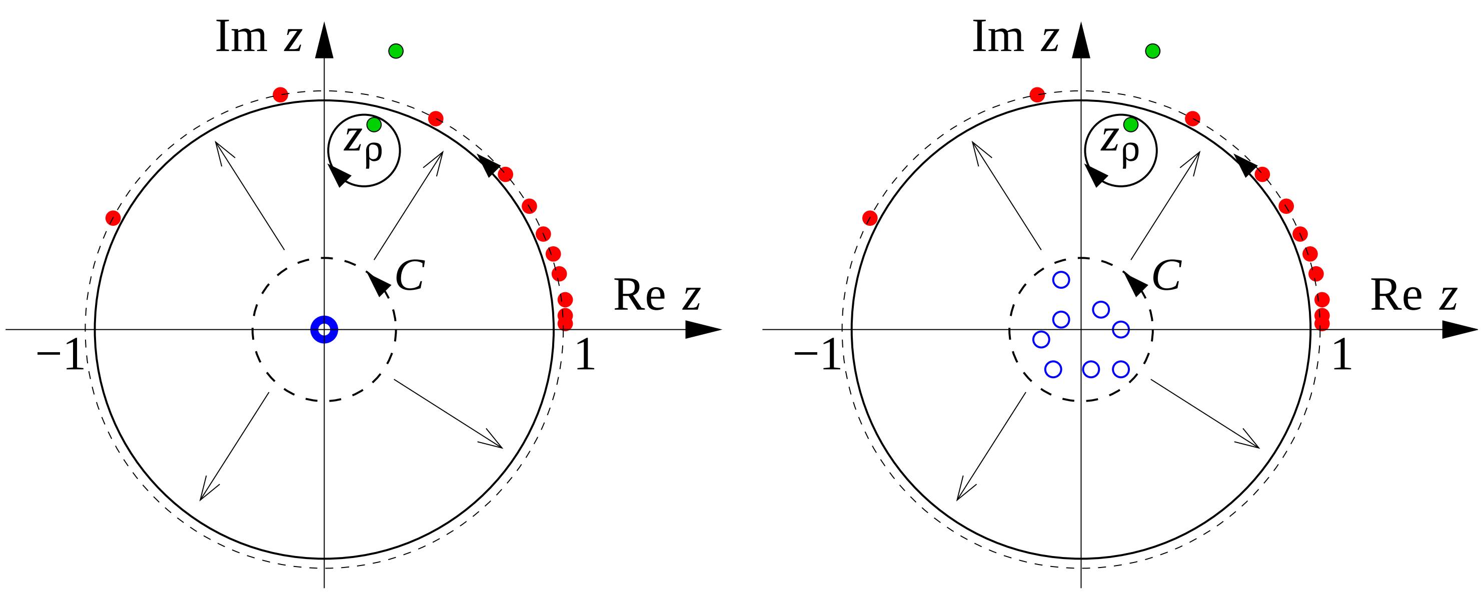
$$\lambda_n = \frac{1}{2\pi \mathrm{i}}\oint_C \frac{\mathrm{d}z}{z^n}\,F(x(z)), \iff$$

Denominator: z^n $\mapsto$ New denominator $(z-z_1)\cdots(z-z_n)$



Hyperbolic translation (Möbius transformation): $z \mapsto H_{\tilde z}(z) = \dfrac{z - \tilde z}{1 - \tilde z^* z}$

$$\frac{1}{n}\lambda_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}z}{z^{n+1}} F(x(z)), \qquad F(x(z)) \stackrel{\text{def}}{=} \frac{\mathrm{d}}{\mathrm{d}z} \log 2\xi\left(\frac{1}{1-z}\right)$$

$$\downarrow$$

$$\Lambda_n = \frac{1}{2\pi \mathrm{i}} \oint_C \frac{\mathrm{d}z}{z\, H_{z_1}(z) \cdots H_{z_n}(z)} F(x(z)) \equiv \sum_{m=1}^{n} \frac{1}{z_m [H_{z_1}(z) \cdots H_{z_n}(z)]'(z_m)} F_m$$

with $F_m \stackrel{\text{def}}{=} F(x(z_m))$, giving an explicit *finite-difference* formula.

Finally, under the choice $z_m = 1 - (2m)^{-1}$,

$$\boxed{\Lambda_n = (-1)^n \sum_{m=1}^{n} (-1)^m \frac{2^{m-n}\,(2(n+m)-1)!!}{(2m-1)\,(n-m)!\,(2m)!} \log\left[\frac{|B_{2m}|}{(2m-3)!!}\,(2\pi)^m\right]}$$

using $F_m = \log 2\xi(2m)$ (also explicit).

Numerical computation of the Λ_n

```
Mathematica 10.3.0 for Linux x86 (64-bit)
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In[1]:= p:=Floor[0.76555 n] + 16

In[2]:= n=1000

Out[2]= 1000

In[3]:= Sum[(-1)^m N[ (2(m+n)-1)!! / (m! (n-m)! (2m-1)!! (2m-1))
        (Log[ Abs[ BernoulliB[2m]]]-Log[(2m-3)!!] ),p],{m,n}] /(-2)^n +
        N[ (1-(-2)^n n!/(2n-1)!!) Log[2 Pi]/2, p]

Out[3]= 6.12442233724777030
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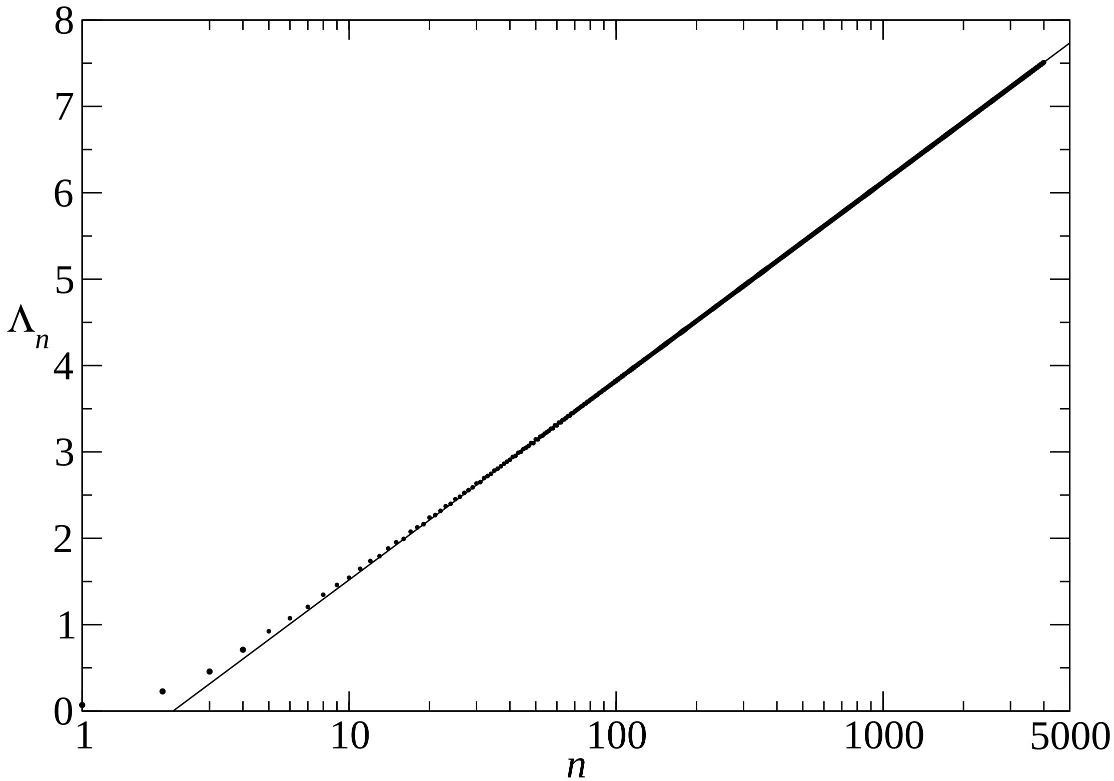
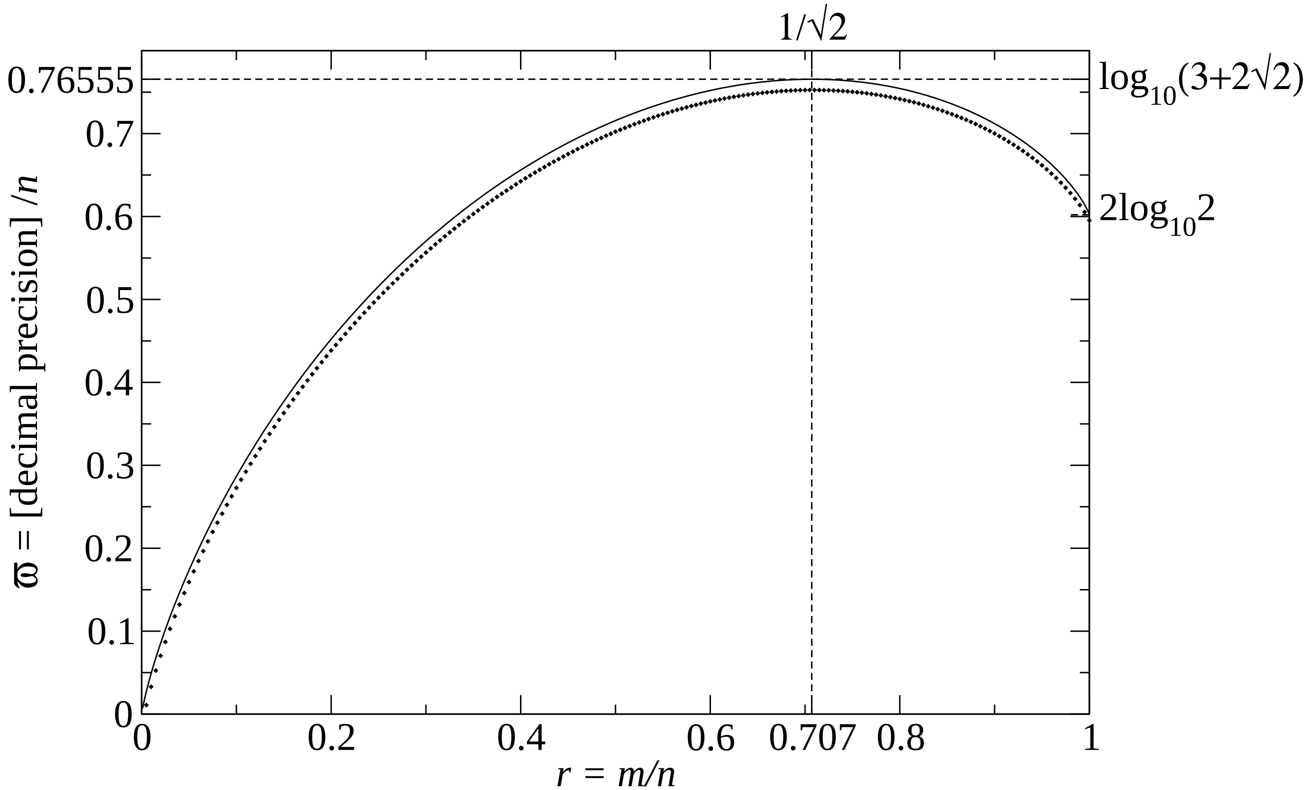


Λ_n computed up to $n = 4000$, on a logarithmic n-scale; straight line: the asymptotic form $\bar{\Lambda}_n = \log n + \frac{1}{2}(\gamma - \log \pi - 1) \approx \log n - 0.78375711$.

$$\Lambda_1 = \frac{3}{2} \log \frac{\pi}{3} \approx 0.0691764, \ \Lambda_2 \approx 0.2274543, \ \Lambda_3 \approx 0.4567141$$

Adjustable precision $p(m) \approx \log_{10}|s_{nm}| + D$

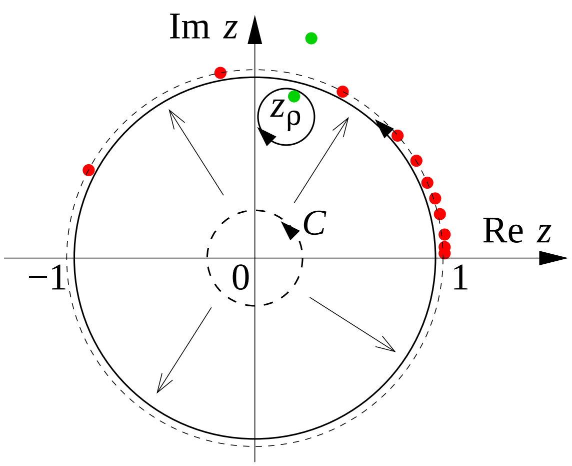
```
In[1]:= p:=Floor[0.76555 n] + 16
```

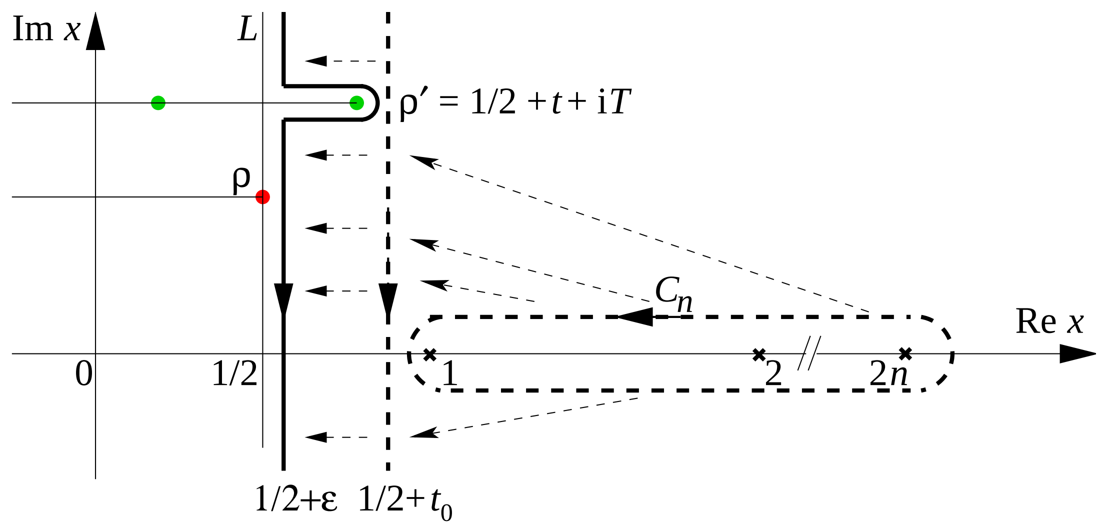


$$\log_{10}|s_{nm}| \sim \left[-2\,\frac{m}{n}\log_{10}\frac{m}{n}+\left(1+\frac{m}{n}\right)\log_{10}\left(1+\frac{m}{n}\right)-\left(1-\frac{m}{n}\right)\log_{10}\left(1-\frac{m}{n}\right)\right]n.$$

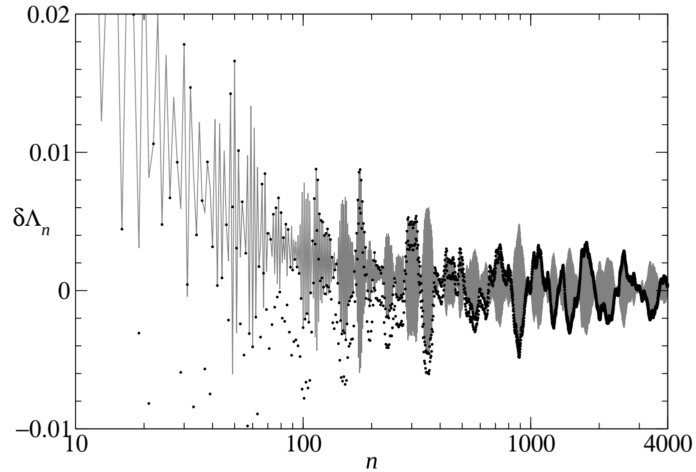
Asymptotic alternative (AV 2015)

$$\Lambda_n \sim \begin{cases} \displaystyle\sum_{\{\mathrm{Re}\ \rho > 1/2\}} e^{i\phi(\rho)} \frac{(-1)^n (2n)^{\mathrm{Re}\ \rho - 1/2}}{|\mathrm{Im}\ \rho|^{\mathrm{Re}\ \rho + 3/2} \log n} & \textbf{if RH false} \\ \textit{(power-like oscillations of both signs)} & \\ \log n + \frac{1}{2} n(\gamma - \log \pi - 1) \quad + o(1) & \textbf{if RH true} \\ \textit{(tempered growth to } +\infty\textit{)} & \end{cases}$$

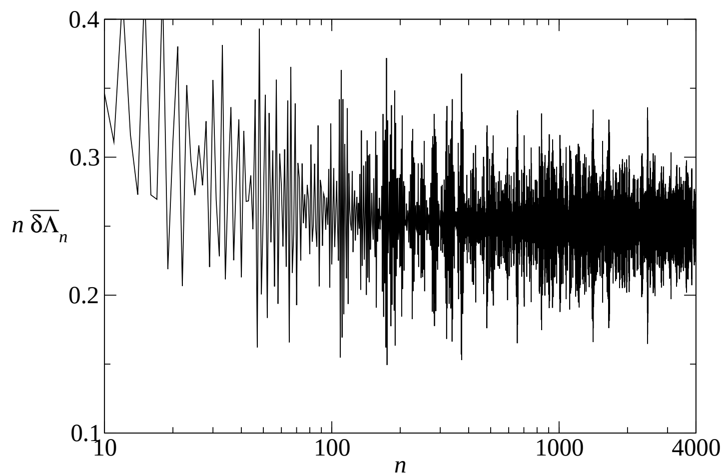




$$\lambda_n = \frac{1}{2\pi i} \oint_C \frac{F}{z^n} \mathrm{d}z \sim - \sum_{\{|z_\rho| < 1\}} z_\rho^{-n}; \ \Lambda_n \sim \frac{1}{2\pi i} \oint_{C_n} G\, n^{x - 1/2} \mathrm{d}x \sim \sum_{\{\mathrm{Re}\ \rho > 1/2\}} e^{i\phi(\rho)} \frac{(-1)^n (2n)^t}{|T|^{t+2} \log n}$$



The remainder sequence $\delta\Lambda_n = \Lambda_n - \bar{\Lambda}_n$ (in gray), and a rectified form $(-1)^n \delta\Lambda_n$ (black dots) to cancel the period-2 oscillations.



Averaged and rescaled remainder sequence $n\,\overline{\delta\Lambda_n} \stackrel{\text{def}}{=} \frac{1}{2}n(\delta\Lambda_n + \delta\Lambda_{n-1})$.