Partial Bergman kernels and the quantum Hall effect

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Bergman kernels on positive line bundles

This talk concerns the space $H^0(M, L^k)$ of holomorphic sections of the kth power of a positive Hermitian holomorphic line bundle $L \to M$ over a Kähler manifold (M, ω). The Hermitian metric is denoted by h and in a local frame e_L it is denoted by $|e_L(z)|_h^2 = e^{-\varphi}$. Positive Hermitian means that $i\partial\bar{\partial} \log h = \omega$ is a Kähler form. The key object in this talk is the orthogonal

projection,

$$\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$$

with respect to the inner product

$$\langle s_1, s_2 \rangle := \int_M (s_1(z), s_2(z))_{h^k} \frac{\omega^m}{m!}.$$

The Schwartz kernel of $\Pi_{h^k}(z, w)$ relative to the volume form $\frac{\omega^m}{m!}$ is known as the semi-classcial Bergman kernel or Szego kernel.

Boutet de Monvel-Sjostrand parametrix

The projections Π_{h^k} onto $H^0(M, L^k)$ lift to projections $\hat{\Pi}_{h_k}$ on the principal S^1 bundle $\partial D_h^* \subset L^*$ where $D_h^* = \{(z, \lambda) : |\lambda|_{h_z} < 1\}$. This is a strictly pseudo-convex domain in L^*. The sum $\Pi = \sum_{k \geq 0} \hat{\Pi}_{h_k}$ is the true Szego kernel

$$\hat{\Pi} : L^2(\partial D_h^*) \to H^2(\partial D_h^*)$$

onto boundary values of holomorphic functions on D_h^*

Near the diagonal in $\partial D_h^* \times \partial D_h^*$, the Boutet de Monvel-Sjostrand parametrix is:

$$\hat{\Pi}(x, y) = \int_0^\infty e^{-\sigma \psi(x,y)} \chi(x, y) s(x, y, \sigma) d\sigma + \hat{R}(x, y). \quad (1)$$

Here, $\chi(x, y)$ is a smooth cutoff to the diagonal; $s(x, y, \sigma)$ is a semi-classical symbol of order $m = \dim_{\mathbb{C}} M$.

The phase

When the Kähler metric ω is real analytic, the phase ψ is constructed from the Kähler potential $\varphi(z)$ of ω_0 by

$$\psi(x, y) = \psi((z, \lambda), (w, \mu)) = 1 - \lambda\bar{\mu}e^{\varphi(z,\bar{w})} \tag{2}$$

where $\varphi(z, \bar{w})$ is the analytic extension of $\varphi(z) = \varphi(z, \bar{z})$ into the complexification $M \times \bar{M}$ of M. Also,

$$s \sim \sum_{n=0}^{\infty} \sigma^{m-n} s_n(x, y) \tag{3}$$

is an analytic symbol in the sense of Boutet de Monvel. Finally, the remainder term $\hat{R}(x, y)$ is real analytic in a neighborhood of the diagonal. If ω is only C^∞ then $\psi(z, w)$ is defined by an almost-analytic extension and the remainder R is C^∞.

Partial Bergman kernels

Our interest is not in the full Bergman kernel but in the partial Bergman kernels (PBK's): Partial Bergman kernels

$$\Pi_{k,\mathcal{S}_k} : L^2(M, L^k) \to \mathcal{S}_k \subset H^0(M, L^k) \tag{4}$$

are orthogonal projections onto proper subspaces $\mathcal{S}_k$ of the holomorphic sections of L^k. For certain sequences $\mathcal{S}_k$ of subspaces, the partial density of states $k^{-m}\Pi_{k,\mathcal{S}_k}(z,z)$ has an asymptotic expansion as $k \to \infty$ which roughly gives the probability density that a quantum state from $\mathcal{S}_k$ is at the point z. If $\{s_{k,j}\}$ is an ONB for $\mathcal{S}_k$, then

$$\Pi_{k,\mathcal{S}_k}(z,z) = \sum_{j=1}^{\dim \mathcal{S}_k} |s_{k,j}(z)|^2_{h^k}.$$

Here and henceforth, the value on the diagonal means the metric contraction.

Sections vanishing to high order on a hypersurface

A motivating example: Let $Y \subset M$ be a complex hypersurface (divisor). Let

$$\mathcal{S}_{k,t}^{Y} := H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}),$$

holomorphic sections vanishing to order tk on Y.
Define the orthogonal projections

$$\Pi_k^{Y,t}(z,w) : L^2(X, L^k) \to H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}). \qquad (5)$$

Can one find the asymptotics $\Pi_k^{Y,t}(z,z)$?

It is "obvious" that $\Pi_k^{Y,t}(z,z)$ should be exponentially decaying on Y and in *some* tubular neighborhood of Y. But the details are not known except in special cases (Toric (M, L, h)).

A motivating problem from the QHE

Other examples of Partial Bergman kernels arise in the quantum Hall effect.
Suppose $\Omega \subset M$ is a domain in a kahler manifold M^m of dimension m. We would like to fill it up with quantum states from $H^0(M, L^k)$, with no 'spill-over' into $M \backslash \Omega$. If the states are $\{s_j^k\}_{j=1}^{d_{k,\Omega}}$ then heuristically we want

$$k^{-m} \sum_{j=1}^{d_{k,\Omega}} |s_j^k(z)|_{h^k} \simeq C_m k^m \mathbf{1}_\Omega(z).$$

Here, $\mathbf{1}_\Omega$ is the characteristic function of Ω. Also, $d_{k,\Omega}$ is the dimension of the relevant subspace.

Of course, this is not literally possible. How close can we come? What does the minimal 'spill-over look like".

Spectral theory of Toeplitz operators

If $\hat{H}_k : H^0(M, L^k) \to H^0(M, L^k)$ is a self-adjoint Toeplitz operator such as $\hat{H}_k = \Pi_{h^k} H \Pi_{h^k}$, then one might define $\mathcal{S}_k$ to be a spectral subspace of $\hat{H}_k$. In terms of

$$\hat{H}_k s_{k,j} = \mu_{k,j} s_{k,j}$$

one may define

$$\mathcal{S}_k = \mathrm{Span}\{\mathrm{s_{k,j}} : \mu_{\mathrm{k,j}} \in [\mathrm{E_1}, \mathrm{E_2}]\}.$$

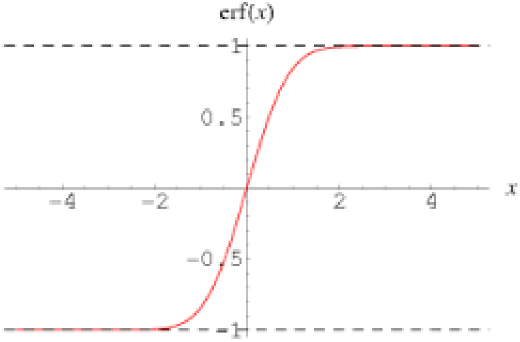
The corresponding partial Bergman kernel is the orthogonal projection

$$\Pi_{k,\mathcal{S}_k} = \mathbf{1}_{[E_1,E_2]}(\hat{H}_k)$$

to this subspace.

The answer in the S^1-invariant 1D case

The 'limit shape' of the interface is an complete Gaussian:



There is an *allowed region* where the PBK is almost 1 and the *forbidden region* where it is almost zero. The transition region has width $O(\frac{1}{\sqrt{k}})$. This picture is in most standard texts on QHE.

Toric Kähler manifolds

The simplest partial Bergman kernels arise from toric Kähler manifolds M^m. Then $H^0(M, L^k)$ is spanned by monomials z^α where $\alpha \in kP \cap \mathbb{Z}^m$ is a lattice point in the kth dilate of the polytope P corresponding to M, i.e. the image

$$\mu : M \to P$$

under the moment map. Subspaces may be defined by choosing sub-polytopes $P' \subset P$ which are 'Delzant'. The corresponding z^α's vanish to high order on the divisor at infinity.

- Shiffman -Zelditch (2004) In the *allowed region* $\mathcal{A} := \mu^{-1}(P')$, the PBK asymptotics are the same as for the full Bergman kernel. In the *forbidden region* $\mathcal{F} := M \backslash \mu^{-1}(P')$, they are exponentially decaying. The decay rate is an explicit Agmon type function $b_{P'}$.
- (2014) Pokorny-Singer: Generalized the allowed asymptotics of (Sh-Z) to any toric Kahler manifold and toric divisor. Main novelty: distributional expansion of PBK on $\partial \mathcal{A}$.

Density of states for a toric sub-polytope PBK

Figure : Gaussian decay from allowed to forbidden

S^1-invariant Kähler manifolds

Ross-Singer generalized the toric results to Kähler manifolds with a holomorphic Hamiltonian S^1 symmetry , with a hypersurface $Y \subset M^{S^1}$ contained in the fixed point set (critical point set of the Hamiltonian).

- (2014) Ross-Singer: Discovered the incomplete Gaussian interface asymptotics for S^1-invariant (M, L, h) for PBK's onto sections vanishing to order tk on an S^1-invariant hypersurface Y.

- (2016) (Peng Zhou-S. Z.) General Hamiltonian S^1-invariant (M, L, h) with no assumptions on fixed point set; exponential decay rate and interface asymptotics.

  Many of the results (in particular the interface asymptotics) are valid for any Hamiltonian (in progress).

Filling domains with quantum states

If the domain Ω has the form $E_1 \leq H \leq E_2$ for some $H : M \to \mathbb{R}$, then the eigensections (i.e. partial Bergman kernel) for the spectral subspace of $\hat{H}_k$ with eigenvalues in $[E_1, E_2]$ will fill it.

Another approach is to study the spectral theory of $\Pi_{h^k} \mathbf{1}_\Omega \Pi_{h^k}$. R. Berman proved a Szego limit theorem for the eigenvalues of this operator. No results yet on interface asymptotics.

Shiffman-S.Z. and P. Zhou- S.Z. use the Boutet-de-Monvel-Sjostrand parametrix to obtain simple and accurate results.

Hamiltonian-holomorphic S^1 actions on Kaehler manifolds

We now describe the results in the S^1 case. The seting consists of a positive Hermitian holomorphic line bundle $(L, h) \to (M, \omega, J)$ over a Kähler manifold of complex dimension m carrying a Hamiltonian holomorphic S^1 action

$$\exp i\theta\, \xi_H : \mathbb{T} \times M \to M, \quad \iota_{\xi_H}\omega = dH$$

where $H : M \to \mathbb{R}$ is the Hamiltonian. Here $\mathbb{T} = S^1$.

Holomorphic means that for each θ, $\exp i\theta\, \xi_H$ is a holomorphic map. It follows that each is an isometry of $g(X, Y) = \omega(X, JY)$.

It was observed by T. Frankel that a holomorphic S^1 action is always Hamiltonian if its fixed point set is non-empty.

Model examples on $\mathbb{C}^{m+1}$.

Standard S^1 actions on $\mathbb{C}^{m+1}$ have the form

$$e^{i\theta} \cdot [Z_0, \dots, Z_m] = [e^{ib_0\theta} Z_0, \dots, e^{ib_m\theta} Z_m], \quad b_j \in \mathbb{Z}.$$

Extreme cases:

$$(i)\ e^{i\theta} \cdot [Z_0, \dots, Z_m] = [e^{i\theta} Z_0, Z_1, \dots, Z_m],\ \mathrm{Hamiltonian}\ |\mathrm{Z_0}|^2,$$

with fixed point manifold $Z_0 = 0$;

$$(ii)\ e^{i\theta} \cdot [Z_0, \dots, Z_m] = [e^{i\theta} Z_0, \dots, e^{i\theta} Z_m],\ \ \mathrm{Hamiltonian}\ \sum_{\mathrm{j=1}}^{\mathrm{m}} |\mathrm{Z_j}|^2$$

with isolated fixed point $\{0\}$.

Model examples on $\mathbb{CP}^m$

Standard S^1 actions on $\mathbb{CP}^m$ arise from subgroups $S^1 \subset SU(m+1)$ of the form

$$e^{i\theta} \cdot [Z_0, \dots, Z_m] = [Z_0, e^{ib_1\theta} Z_1, \dots, e^{ib_m\theta} Z_m], \quad b_j \in \mathbb{Z}.$$

With no loss of generality it is assumed that $b_0 = 0$. When $b_j \neq b_k$ when $\mathrm{j} \neq \mathrm{k}$, the action has $m+1$ isolated fixed points, $P_j = [0, \dots, 0, z_j, 0, \dots, 0]$. The weights at P_j are $\{b_j - b_i\}_{j \neq i}$. The Hamiltonian (moment map) is

$$H_{\vec{b}}([Z_0 : \cdots : Z_m]) = \frac{b_1 |Z_1|^2 + \cdots + b_m |Z_m|^2}{|Z|^2}.$$

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S^1 invariant Projective hypersurfaces

Example studied by F. Kirwan: the cubic hypersurface $X \subset \mathbb{CP}^4$,

$$x^3 + y^3 + z^3 = u^2 v,$$

and let $\mathbb{C}^*$ act on X via

$$t \cdot [x, y, z, u, v] = [t^{-1}x, t^{-1}y, t^{-1}z, t^{-3}u, t^3 v].$$

Then $X^{\mathbb{T}}$ has three fixed point components,

$$F_1 = \{[0,0,0,1,0]\}, \quad F_2 = \{[x,y,z,0,0] : x^3 + y^3 + z^3 = 0\},$$

$$F_3 = \{[0,0,0,0,1]\},$$

of which two (F_1, F_3) are isolated fixed points and F_2 is a hypersurface in X, i.e. a curve. The point $P = [0,0,0,0,1]$ is singular.

Ruled surfaces

Another setting of S^1 invariant situation is ruled surfaces.

Let M be a Kähler manifold and let $L \to M$ be a holomorphic line bundle. L carries a natural $\mathbb{C}^*$ action. Projectivize each line $L_z \to \mathbb{P}L_z \simeq \mathbb{CP}^1$ to get $\mathbb{P}L$. It still carries a $\mathbb{C}^*$ action. Equip the $\mathbb{CP}^1$ bundle with an S^1 invariant metric. Then the total space is an S^1-invariant kahler manifold with fixed point components $\simeq M$ corresponding to $0, \infty$ on $\mathbb{CP}^1$.

Linearization (quantization) of the S^1 action

Let h be the Hermitian metric with $i\partial\bar{\partial}\log h = \omega$. If $c_1(L) = [\omega]$, then the Hamiltonian S^1 action preserves (L, h) and can be 'quantized' or linearized to give a representation of $\mathbb{T}$ on the spaces $H^0(X, L^k)$ of holomorphic sections of the tensor powers L^k. The infinitesimal generator acts on a section by

$$\xi \cdot s = (\nabla_\xi + 2\pi ikH)s =: \hat{H}_k s. \qquad (6)$$

Here, ∇ is the Chern connection.

(6) may be integrated to define a unitary representation of $\mathbb{T}$

$$U_k(\theta) = e^{ik\theta\hat{H}_k} : \mathbb{T} \times H^0(M, L^k) \to H^0(M, L^k)$$

on , equipped with the L^2 norm $\mathrm{Hilb}_{\mathrm{h^k}}$ induced by the Hermitian metric h.

Weight decomposition of holomorphic sections and equivariant Bergman kernels

Define the weight spaces by

$$V_k(j) = \{s \in H^0(M, L^k) : U_k(\theta)s = e^{ij\theta}s\}$$
$$= \{s \in H^0(M, L^k) : \hat{H}_k s = \tfrac{j}{k}\, s\}. \tag{7}$$

The associated eigenspace projections

$$\Pi_{k,j}(z, w) : H^0(M, L^k) \to V_k(j) \tag{8}$$

are called "equivariant Bergman kernels". They have been studied in detail by Shiffman-S.Z. (toric), X.Ma-W. Zhang (general compact G), R. Paoletti (general G).

Partial Bergman kernels

The Hamiltonian is a Bott-Morse function $H : M \to [E_-, E_+]$ where $E_\pm = \max / \min H$. Let $P \subset (E_-, E_+)$ be a proper closed interval. Define the corresponding subspace

$$\mathcal{H}_{k,P} := \bigoplus_{j : \frac{j}{k} \in P} V_k(j) \tag{9}$$

and partial Bergman kernels

$$\Pi_{|kP}(z, w) := \sum_{j : \frac{j}{k} \in P} \Pi_{k,j}(z, w). \tag{10}$$

The main problem is to relate the asymptotic properties of $\Pi_{|kP}(z, w)$ to the geometry of $H^{-1}(P)$.

Allowed region, forbidden region and the interface

Define the allowed, resp. forbidden regions by

$$\mathcal{A}_P := \{z \in M : H(z) \in P\}, \quad \mathcal{F}_P := M \backslash \mathcal{A}_P.$$

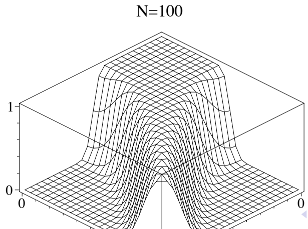
The main idea is that $\Pi_{|kP}(z, z)$ has standard asymptotics in the *allowed region* $\mathcal{A}_P$ and exponentially decaying asymptotics in the forbidden region $\mathcal{F}_P$. The *interface* is

$$\partial \mathcal{A}_P = \partial \mathcal{F}_P.$$

In a special case, Ross-Singer found the scaling limit of $\Pi_{kP}(z, z)$ for z in a transition region between them near the 'interface'.

Allowed vs Forbidden

Allowed: the flat top; Forbidden the flat bottom.



Complexified $\mathbb{C}^*$-action and Agmon distance

The complexification of the holomorphic Hamiltonian $\mathbb{T}$ action is denoted by

$$\tau : \mathbb{C}^* \times M \to M, \quad \tau^*_{e^{i\varphi}} \omega = \omega. \tag{11}$$

The $\mathbb{C}^*$ action is the combined Hamilton flow– gradient flow of the Hamiltonian $H = \mu$ generating the S^1 action.

The exponentially decaying asymptotics in the forbidden region is governed by the 'action' $b_E(z)$ from z to $H^{-1}(E)$. We define b_E by

$$b_E(z) = -E\tau_E(z) + \int_0^{\tau_E(z)} \left[H(e^{-\sigma/2} \cdot z) \right] \cdot d\sigma \,. \tag{12}$$

The integral is over the gradient flow line from z to $H^{-1}(E)$.

Asymptotics of Equivariant Bergman kernels

The following asymptotics are quite simple in the S^1 case:

Theorem

If $|\frac{j}{k} - E| \leq \frac{C \log k}{\sqrt{k}}$, *then*

$$\Pi_{k,j}(z,z) \sim k^{n-1} \frac{1}{\sqrt{\det \varphi''_{\rho\rho}}} A_0 + o(k^{n-1}), \quad z \in H^{-1}(E).$$

For $z \notin E$, *let* $e^{-\tau_E(z)/2} \cdot z \in H^{-1}(E)$. *Then,*

$$\Pi_{k,j}(z,z) \sim k^{n-1} e^{-kb_E(z)} \frac{1}{\sqrt{\det \varphi''_{\rho\rho}}} A_0 + o(k^{n-1}), \quad z \in H^{-1}(E).$$

Here, b_E *is the action integral over the gradient line of* H *from* z *to* $H^{-1}(E)$.

Very simple: just use equivariance. Generalizations to non-abelian groups: X. Ma, R. Paoletti.

Partial Bergman kernel asymptotics

Let $P = [E, H_{\max}]$ where $H_{\max}$ is the maximum value of H or the form $[H_{\min}, E]$ where $H_{\min}$ is the minimum value. It is only notationally more complicated to consider intervals $[E_1, E_2]$ with $E_1 > H_{\min}, E_2 < H_{\max}$.

Theorem

The density of states of the partial Bergman kernel is given by the asymptotic formulas:

$$\Pi_{|kP}(z,z) \sim \begin{cases} c_0 + c_1 k^{-1} + c_2 k^{-2} + \cdots, & \text{for } z \in H^{-1}(P), \\ k^{-m} e^{-k b_E(z)} \left[c_0(z) + O(k^{-1}) \right], & \text{for } z \in X_1^+, \end{cases}$$

where $c_0 \in \mathcal{C}^\infty(X_1^+)$, and b_E is defined in (12). Furthermore, the remainder estimates are uniform on compact subsets of the basin X_1^+ of attraction of the minimum.

Smoothed partial Bergman kernel asymptotics

The smoothed out interval sums have the form, with $\rho \in \mathcal{S}(\mathbb{R})$,

$$\sum_j \rho(\tfrac{j}{k} - E)\Pi_{k,j}(z,z) = \int_{\mathbb{R}} \hat{\rho}(t) e^{-iE\ t}\Pi_k(e^{it/k}z,z)dt. \qquad (13)$$

THEOREM

$$\sum_j \rho(\frac{j}{k} - E))\Pi_{k,j}(z,z) \sim \begin{cases} c_0 + c_1 k^{-1} + c_2 k^{-2} + \cdots, z \in H^{-1}(P), \\ k^{-m} e^{-k b_E(z)} \left[c_0(z) + O(k^{-1})\right], z \in X_1^+, \end{cases}$$

where the remainder estimates are uniform on compact subsets of the big stratum X_1^+ (big Morse cell).

Interface Asymptotics

The interface asymptotics at a level E involve all of the individual weight Bergman kernels (8) where $|\frac{j}{k} - E| < \frac{C \log k}{k}$.

THEOREM

For z so that $\sqrt{k}(H(z) - \epsilon)$ is bounded, , $k^{-n}\Pi_{|kP}(z,z)$ has a distributional expansion on X whose leading order term is

$$k^{-n}\Pi_{|kP}(z,z) = \frac{1}{\sqrt{2\pi|\xi_H(z)|^2}} \int_{-\infty}^{\sqrt{k}(H(z)-\epsilon)} e^{-\frac{t^2}{2|\xi_H(z)|^2}} dt + O(k^{-\frac{1}{2}}).$$

ξ_H = Hamilton v.f. of H.

Smoothed interface asymptotics

The smoothed out interface sums have the form, with $\rho \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned} &\sum_j \rho(\sqrt{k}(\tfrac{j}{k} - E))\Pi_{k,j}(z_0 + \tfrac{u}{\sqrt{k}}, z_0 + \tfrac{u}{\sqrt{k}}) \\ &= \int_{\mathbb{R}} \hat{\rho}(t) e^{-iE\sqrt{k}t} \Pi_k(e^{it/\sqrt{k}} z, z) dt. \end{aligned} \tag{14}$$

THEOREM

Let $\mu(z_0) = E$. Then,

$$\sum_j \rho(\sqrt{k}(\tfrac{j}{k} - E))\Pi_{k,j}(z_0 + \tfrac{u}{\sqrt{k}}, z_0 + \tfrac{u}{\sqrt{k}})$$

$$= \int_{\mathbb{R}} \hat{\rho}(t) e^{itE\varphi'_\rho(z_0)u - \varphi''_{\rho\rho}(z_0)t^2} dt + O(\tfrac{1}{\sqrt{k}}).$$

Sketch of proof

Using the Bouet-de-Monvel-Sjostrand parametrix,

$$\sum_j f(\sqrt{k}(\frac{j}{k} - E))\Pi_{k,j}(z,z) = \int_{\mathbb{R}} \hat{f}(t)e^{-iE\sqrt{k}t}\Pi_k(e^{it/\sqrt{k}}z,z)dt$$

$$= k^m \int_{-\infty}^{\infty} \hat{f}(t)e^{-it(\sqrt{k}E)}e^{k\psi(e^{it/(2\sqrt{k})}\cdot z, e^{-it/(2\sqrt{k})}\cdot z) - k\varphi(z)} A_k(e^{it/2\sqrt{k}}z,z)\frac{dt}{2\pi}$$

$$+ k^m \int_{-\infty}^{\infty} \hat{f}(t)e^{-it(\sqrt{k}E)}R_k(e^{it/2\sqrt{k}}z,z)\frac{dt}{2\pi}$$

Since $R_k \in k^{-\infty}C^\infty(M \times M)$, the second term is $O(k^{-\infty})$. We note that $t \to \Pi_k(e^{it/\sqrt{k}}z,z)$ is $2\pi\sqrt{k}$-periodic (similarly for the parametrix and remainder terms), so the integrals converge when $\hat{f} \in L^1(\mathbb{R}^d)$.

Continuation

Let

$$\Psi(\tau, z) = -\tau(\sqrt{k}E) + k\psi(e^{\tau/2\sqrt{k}} \cdot z, e^{\bar{\tau}/2\sqrt{k}} \cdot z) - k\varphi(z),$$

so the phase is $\Psi(it)$. If $\varphi(z)$ is real analytic, then $\Psi(\tau)$ is holomorphic when $\Im(\tau)$ is small enough. If φ is only smooth, then $\Psi(\tau)$ is an almost analytic extension of $\Psi|_{\mathbb{R}}$.
If $z = e^{\beta/(\sqrt{k})}z_0$ with $H(z_0) = E$. Then as $k \to \infty$,

$$\Psi(\tau, e^{\beta/(\sqrt{k})}z_0) = -\tau(\sqrt{k}E)+k((\psi(e^{(\tau/2+\beta)/\sqrt{k}}\cdot z_0, e^{(\bar{\tau}/2+\beta)/\sqrt{k}}\cdot z_0)$$

$$-\varphi(e^{\beta/\sqrt{k}} \cdot z_0)$$

$$= \frac{1}{2}((\tau/2 + \beta)^2 - \beta^2)\partial_\rho^2\varphi(z_0) + g_3(z, \tau, \beta),$$

where

$$g_3 = O(k^{-1/2}(|\beta|^3 + |\tau|^3)).$$

Completion of proof

If $\hat{f} \in C_c(\mathbb{R})$, using the Plancherel theorem and the Taylor expansion above, the PBK is

$$k^m \int_{-\infty}^{\infty} \hat{f}(t) \left[e^{\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial_\rho^2\varphi(z_0)} e^{g_3} dt \right] (1 + O(k^{-1}))$$

$$= k^m \int_{-\infty}^{\infty} \hat{f}(t) \left[e^{\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial_\rho^2\varphi(z_0)} dt \right] + O(k^{m-\frac{1}{2}}))$$

$$= k^m \int_{-\infty}^{\infty} f(x) \left[\int_{-\infty}^{\infty} e^{-itx+\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial_\rho^2\varphi(z_0)} dt \right] \frac{dx}{2\pi} + O(k^{m-\frac{1}{2}}))$$

$$= k^m \int_{-\infty}^{\infty} f(x) \sqrt{\frac{2}{\pi \partial_\rho^2 \varphi(z_0)}} e^{-\frac{(2x-\beta\partial_\rho^2\varphi(z_0))^2}{2\partial_\rho^2\varphi(z_0)}} dx + O(k^{m-1/2}))$$