Partial Bergman kernels and the quantum Hall effect

Steve Zelditch; joint work with Peng Zhou

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Bergman kernels on positive line bundles

This talk concerns the space $H^0(M, L^k)$ of holomorphic sections of the kth power of a positive Hermitian holomorphic line bundle $L \to M$ over a Kähler manifold $(M, \omega)$. The Hermitian metric is denoted by $h$ and in a local frame $e_L$ it is denoted by $|e_L(z)|^2_h = e^{-\varphi}$. Positive Hermitian means that $i\partial \bar{\partial} \log h = \omega$ is a Kähler form. The key object in this talk is the orthogonal projection,

$$\Pi_{h^k} : L^2(M, L^k) \to H^0(M, L^k)$$

with respect to the inner product

$$\langle s_1, s_2 \rangle := \int_M (s_1(z), s_2(z))_{h^k} \frac{\omega^m}{m!}.$$

The Schwartz kernel of $\Pi_{h^k}(z, w)$ relative to the volume form $\frac{\omega^m}{m!}$ is known as the semi-classical Bergman kernel or Szego kernel.
The projections $\Pi_{h^k}$ onto $H^0(M, L^k)$ lift to projections $\hat{\Pi}_{h^k}$ on the principal $S^1$ bundle $\partial D_h^* \subset L^*$ where $D_h^* = \{(z, \lambda) : |\lambda|_{h_z} < 1\}$. This is a strictly pseudo-convex domain in $L^*$. The sum $\Pi = \sum_{k \geq 0} \hat{\Pi}_{h^k}$ is the true Szego kernel

$$\hat{\Pi} : L^2(\partial D^*_h) \to H^2(\partial D^*_h)$$

to boundary values of holomorphic functions on $D^*_h$

Near the diagonal in $\partial D^*_h \times \partial D^*_h$, the Boutet de Monvel-Sjostrand parametrix is:

$$\hat{\Pi}(x, y) = \int_{0}^{\infty} e^{-\sigma \psi(x, y)} \chi(x, y) s(x, y, \sigma) d\sigma + \hat{R}(x, y). \quad (1)$$

Here, $\chi(x, y)$ is a smooth cutoff to the diagonal; $s(x, y, \sigma)$ is a semi-classical symbol of order $m = \dim_{\mathbb{C}} M$. 
The phase

When the Kähler metric $\omega$ is real analytic, the phase $\psi$ is constructed from the Kähler potential $\varphi(z)$ of $\omega_0$ by

$$
\psi(x, y) = \psi((z, \lambda), (w, \mu)) = 1 - \lambda \mu e^{\varphi(z, \bar{w})}
$$

(2)

where $\varphi(z, \bar{w})$ is the analytic extension of $\varphi(z) = \varphi(z, \bar{z})$ into the complexification $M \times \bar{M}$ of $M$. Also,

$$
s \sim \sum_{n=0}^{\infty} \sigma^{m-n} s_n(x, y)
$$

(3)

is an analytic symbol in the sense of Boutet de Monvel. Finally, the remainder term $\hat{R}(x, y)$ is real analytic in a neighborhood of the diagonal. If $\omega$ is only $C^\infty$ then $\psi(z, w)$ is defined by an almost-analytic extension and the remainder $R$ is $C^\infty$. 
Partial Bergman kernels

Our interest is not in the full Bergman kernel but in the partial Bergman kernels (PBK’s): Partial Bergman kernels

\[ \Pi_{k,S_k} : L^2(M, L^k) \to S_k \subset H^0(M, L^k) \]  

(4)

are orthogonal projections onto proper subspaces \( S_k \) of the holomorphic sections of \( L^k \). For certain sequences \( S_k \) of subspaces, the partial density of states \( k^{-m}\Pi_{k,S_k}(z, z) \) has an asymptotic expansion as \( k \to \infty \) which roughly gives the probability density that a quantum state from \( S_k \) is at the point \( z \). If \( \{s_{k,j}\} \) is an ONB for \( S_k \), then

\[ \Pi_{k,S_k}(z, z) = \sum_{j=1}^{\dim S_k} |s_{k,j}(z)|^2_{h_k}. \]

Here and henceforth, the value on the diagonal means the metric contraction.
Sections vanishing to high order on a hypersurface

A motivating example: Let $Y \subset M$ be a complex hypersurface (divisor). Let

$$S_{k,t}^Y := H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}),$$

holomorphic sections vanishing to order $tk$ on $Y$.

Define the orthogonal projections

$$\Pi_{k,t}^Y(z, w) : L^2(X, L^k) \rightarrow H^0(X, \mathcal{O}(L^k) \otimes \mathcal{I}_Y^{tk}). \quad (5)$$

Can one find the asymptotics $\Pi_{k,t}^Y(z, z)$?

It is “obvious” that $\Pi_{k,t}^Y(z, z)$ should be exponentially decaying on $Y$ and in some tubular neighborhood of $Y$. But the details are not known except in special cases (Toric $(M, L, h)$).
A motivating problem from the QHE

Other examples of Partial Bergman kernels arise in the quantum Hall effect. Suppose $\Omega \subset M$ is a domain in a kahler manifold $M^m$ of dimension $m$. We would like to fill it up with quantum states from $H^0(M, L^k)$, with no ‘spill-over’ into $M \setminus \Omega$. If the states are $\{ s_j^k \}^{d_k, \Omega}_{j=1}$ then heuristically we want

$$ k^{-m} \sum_{j=1}^{d_k, \Omega} |s_j^k(z)|_{h^k} \simeq C_m k^m \mathbf{1}_\Omega(z). $$

Here, $\mathbf{1}_\Omega$ is the characteristic function of $\Omega$. Also, $d_{k, \Omega}$ is the dimension of the relevant subspace.

Of course, this is not literally possible. How close can we come? What does the minimal ‘spill-over’ look like”.
Spectral theory of Toeplitz operators

If $\hat{H}_k : H^0(M, L^k) \to H^0(M, L^k)$ is a self-adjoint Toeplitz operator such as $\hat{H}_k = \Pi_{h^k} H \Pi_{h^k}$, then one might define $S_k$ to be a spectral subspace of $\hat{H}_k$. In terms of $\hat{H}_k s_{k,j} = \mu_{k,j} s_{k,j}$

one may define

$$S_k = \text{Span}\{s_{k,j} : \mu_{k,j} \in [E_1, E_2]\}.$$

The corresponding partial Bergman kernel is the orthogonal projection

$$\Pi_{k,S_k} = 1_{[E_1, E_2]}(\hat{H}_k)$$

to this subspace.
The answer in the $S^1$-invariant 1D case

The ‘limit shape’ of the interface is an complete Gaussian:

There is an allowed region where the PBK is almost 1 and the forbidden region where it is almost zero. The transition region has width $O\left(\frac{1}{\sqrt{k}}\right)$. This picture is in most standard texts on QHE.
Toric Kähler manifolds

The simplest partial Bergman kernels arise from toric Kähler manifolds $M^m$. Then $H^0(M, L^k)$ is spanned by monomials $z^\alpha$ where $\alpha \in kP \cap \mathbb{Z}^m$ is a lattice point in the kth dilate of the polytope $P$ corresponding to $M$, i.e. the image

$$\mu : M \rightarrow P$$

under the moment map. Subspaces may be defined by choosing sub-polytopes $P' \subset P$ which are ‘Delzant’. The corresponding $z^{\alpha}$’s vanish to high order on the divisor at infinity.

- Shiffman -Zelditch (2004) In the allowed region $A := \mu^{-1}(P')$, the PBK asymptotics are the same as for the full Bergman kernel. In the forbidden region $F := M \setminus \mu^{-1}(P')$, they are exponentially decaying. The decay rate is an explicit Agmon type function $b_{P'}$.

- (2014) Pokorny-Singer: Generalized the allowed asymptotics of (Sh-Z) to any toric Kahler manifold and toric divisor. Main novelty: distributional expansion of PBK on $\partial A$. 
Density of states for a toric sub-polytope PBK

Figure: Gaussian decay from allowed to forbidden
$S^1$-invariant Kähler manifolds

Ross-Singer generalized the toric results to Kähler manifolds with a holomorphic Hamiltonian $S^1$ symmetry, with a hypersurface $Y \subset M^{S^1}$ contained in the fixed point set (critical point set of the Hamiltonian).

- (2014) Ross-Singer: Discovered the incomplete Gaussian interface asymptotics for $S^1$-invariant $(M, L, h)$ for PBK’s onto sections vanishing to order $tk$ on an $S^1$-invariant hypersurface $Y$.

- (2016) (Peng Zhou-S. Z.) General Hamiltonian $S^1$-invariant $(M, L, h)$ with no assumptions on fixed point set; exponential decay rate and interface asymptotics.

Many of the results (in particular the interface asymptotics) are valid for any Hamiltonian (in progress).
Filling domains with quantum states

If the domain $\Omega$ has the form $E_1 \leq H \leq E_2$ for some $H : M \to \mathbb{R}$, then the eigensections (i.e. partial Bergman kernel) for the spectral subspace of $\hat{H}_k$ with eigenvalues in $[E_1, E_2]$ will fill it.

Another approach is to study the spectral theory of $\Pi_{hk}1_\Omega \Pi_{hk}$. R. Berman proved a Szego limit theorem for the eigenvalues of this operator. No results yet on interface asymptotics.

Shiffman-S.Z. and P. Zhou- S.Z. use the Boutet-de-Monvel-Sjostrand parametrix to obtain simple and accurate results.
Hamiltonian-holomorphic $S^1$ actions on Kaehler manifolds

We now describe the results in the $S^1$ case. The setting consists of a positive Hermitian holomorphic line bundle $(L, h) \rightarrow (M, \omega, J)$ over a Kähler manifold of complex dimension $m$ carrying a Hamiltonian holomorphic $S^1$ action

$$\exp i \theta \xi_H : \mathbb{T} \times M \rightarrow M, \quad \iota_{\xi_H} \omega = dH$$

where $H : M \rightarrow \mathbb{R}$ is the Hamiltonian. Here $\mathbb{T} = S^1$.

Holomorphic means that for each $\theta$, $\exp i \theta \xi_H$ is a holomorphic map. It follows that each is an isometry of $g(X, Y) = \omega(X, JY)$.

It was observed by T. Frankel that a holomorphic $S^1$ action is always Hamiltonian if its fixed point set is non-empty.
Model examples on $\mathbb{C}^{m+1}$.

Standard $S^1$ actions on $\mathbb{C}^{m+1}$ have the form

$$e^{i\theta} \cdot [Z_0, \ldots, Z_m] = [e^{ib_0 \theta} Z_0, \ldots, e^{ib_m \theta} Z_m], \quad b_j \in \mathbb{Z}.$$ 

Extreme cases:

(i) $e^{i\theta} \cdot [Z_0, \ldots, Z_m] = [e^{i\theta} Z_0, Z_1, \ldots, Z_m]$, Hamiltonian $|Z_0|^2$, with fixed point manifold $Z_0 = 0$;

(ii) $e^{i\theta} \cdot [Z_0, \ldots, Z_m] = [e^{i\theta} Z_0, \ldots, e^{i\theta} Z_m]$, Hamiltonian $\sum_{j=1}^{m} |Z_j|^2$ with isolated fixed point $\{0\}$. 
Model examples on $\mathbb{C}P^m$

Standard $S^1$ actions on $\mathbb{C}P^m$ arise from subgroups $S^1 \subset SU(m+1)$ of the form

$$e^{i\theta} \cdot [Z_0, \ldots, Z_m] = [Z_0, e^{ib_1 \theta} Z_1, \ldots, e^{ib_m \theta} Z_m], \quad b_j \in \mathbb{Z}.$$  

With no loss of generality it is assumed that $b_0 = 0$. When $b_j \neq b_k$ when $j \neq k$, the action has $m+1$ isolated fixed points, $P_j = [0, \ldots, 0, z_j, 0, \ldots, 0]$. The weights at $P_j$ are $\{b_j - b_i\}_{j \neq i}$. The Hamiltonian (moment map) is

$$H_b([Z_0 : \cdots : Z_m]) = \frac{b_1 |Z_1|^2 + \cdots + b_m |Z_m|^2}{|Z|^2}.$$
$S^1$ invariant Projective hypersurfaces

Example studied by F. Kirwan: the cubic hypersurface $X \subset \mathbb{CP}^4$, 

$$x^3 + y^3 + z^3 = u^2 v,$$

and let $\mathbb{C}^*$ act on $X$ via 

$$t \cdot [x, y, z, u, v] = [t^{-1}x, t^{-1}y, t^{-1}z, t^{-3}u, t^3 v].$$

Then $X^\mathbb{T}$ has three fixed point components, 

$$F_1 = \{[0, 0, 0, 1, 0]\}, \quad F_2 = \{[x, y, z, 0, 0] : x^3 + y^3 + z^3 = 0\},$$

$$F_3 = \{[0, 0, 0, 0, 1]\},$$

of which two ($F_1, F_3$) are isolated fixed points and $F_2$ is a hypersurface in $X$, i.e. a curve. The point $P = [0, 0, 0, 0, 1]$ is singular.
Another setting of $S^1$ invariant situation is ruled surfaces.

Let $M$ be a Kähler manifold and let $L \to M$ be a holomorphic line bundle. $L$ carries a natural $\mathbb{C}^*$ action. Projectivize each line $L_z \to \mathbb{P}L_z \simeq \mathbb{CP}^1$ to get $\mathbb{P}L$. It still carries a $\mathbb{C}^*$ action. Equip the $\mathbb{CP}^1$ bundle with an $S^1$ invariant metric. Then the total space is an $S^1$-invariant kahler manifold with fixed point components $\simeq M$ corresponding to $0, \infty$ on $\mathbb{CP}^1$. 
Linearization (quantization) of the $S^1$ action

Let $h$ be the Hermitian metric with $i\partial\bar{\partial}\log h = \omega$. If $c_1(L) = [\omega]$, then the Hamiltonian $S^1$ action preserves $(L, h)$ and can be ‘quantized’ or linearized to give a representation of $\mathbb{T}$ on the spaces $H^0(X, L^k)$ of holomorphic sections of the tensor powers $L^k$. The infinitesimal generator acts on a section by

$$\xi \cdot s = (\nabla_\xi + 2\pi ikH)s =: \hat{H}_k s. \quad (6)$$

Here, $\nabla$ is the Chern connection.

(6) may be integrated to define a unitary representation of $\mathbb{T}$

$$U_k(\theta) = e^{ik\theta \hat{H}_k} : \mathbb{T} \times H^0(M, L^k) \to H^0(M, L^k)$$

on $,$ equipped with the $L^2$ norm $\text{Hilb}_{h^k}$ induced by the Hermitian metric $h$. 
Weight decomposition of holomorphic sections and equivariant Bergman kernels

Define the weight spaces by

\[ V_k(j) = \{ s \in H^0(M, L^k) : U_k(\theta)s = e^{ij\theta}s \} \]

\[ = \{ s \in H^0(M, L^k) : \hat{\mathcal{H}}_k s = \frac{j}{k}s \} \].

(7)

The associated eigenspace projections

\[ \Pi_{k,j}(z, w) : H^0(M, L^k) \to V_k(j) \]

(8)

are called “equivariant Bergman kernels”. They have been studied in detail by Shiffman-S.Z. (toric), X.Ma-W. Zhang (general compact $G$), R. Paoletti (general $G$).
Partial Bergman kernels

The Hamiltonian is a Bott-Morse function $H : M \to [E_-, E_+]$ where $E_\pm = \max / \min H$. Let $P \subset (E_-, E_+)$ be a proper closed interval. Define the corresponding subspace

$$\mathcal{H}_{k,P} := \bigoplus_{j : \frac{j}{k} \in P} V_k(j)$$

(9)

and partial Bergman kernels

$$\Pi_{|kP}(z, w) := \sum_{j : \frac{j}{k} \in P} \Pi_{k,j}(z, w).$$

(10)

The main problem is to relate the asymptotic properties of $\Pi_{|kP}(z, w)$ to the geometry of $H^{-1}(P)$. 
Allowed region, forbidden region and the interface

Define the allowed, resp. forbidden regions by

\[ A_P := \{ z \in M : H(z) \in P \}, \quad F_P := M \setminus A_P. \]

The main idea is that \( \Pi_{|kP}(z, z) \) has standard asymptotics in the allowed region \( A_P \) and exponentially decaying asymptotics in the forbidden region \( F_P \). The interface is

\[ \partial A_P = \partial F_P. \]

In a special case, Ross-Singer found the scaling limit of \( \Pi_{kP}(z, z) \) for \( z \) in a transition region between them near the ‘interface’. 
Allowed vs Forbidden

Allowed: the flat top; Forbidden the flat bottom.
Complexified $\mathbb{C}^*$-action and Agmon distance

The complexification of the holomorphic Hamiltonian $\mathbb{T}$ action is denoted by

$$\tau : \mathbb{C}^* \times M \to M, \quad \tau_{e^{i\varphi}}^* \omega = \omega.$$  \hspace{1cm} (11)

The $\mathbb{C}^*$ action is the combined Hamilton flow–gradient flow of the Hamiltonian $H = \mu$ generating the $S^1$ action.

The exponentially decaying asymptotics in the forbidden region is governed by the ‘action’ $b_E(z)$ from $z$ to $H^{-1}(E)$. We define $b_E$ by

$$b_E(z) = -E \tau_E(z) + \int_0^{\tau_E(z)} H(e^{-\sigma/2} \cdot z) \cdot d\sigma.$$  \hspace{1cm} (12)

The integral is over the gradient flow line from $z$ to $H^{-1}(E)$. 
Asymptotics of Equivariant Bergman kernels

The following asymptotics are quite simple in the $S^1$ case:

**Theorem**

If $|\frac{i}{k} - E| \leq \frac{C \log k}{\sqrt{k}}$, then

$$
\Pi_{k,j}(z, z) \sim k^{n-1} \frac{1}{\sqrt{\det \varphi''_{\rho\rho}}} A_0 + o(k^{n-1}), \quad z \in H^{-1}(E).
$$

For $z \notin E$, let $e^{-\tau_E(z)/2} \cdot z \in H^{-1}(E)$. Then,

$$
\Pi_{k,j}(z, z) \sim k^{n-1} e^{-kb_E(z)} \frac{1}{\sqrt{\det \varphi''_{\rho\rho}}} A_0 + o(k^{n-1}), \quad z \in H^{-1}(E).
$$

Here, $b_E$ is the action integral over the gradient line of $H$ from $z$ to $H^{-1}(E)$.

Partial Bergman kernel asymptotics

Let $P = [E, H_{\text{max}}]$ where $H_{\text{max}}$ is the maximum value of $H$ or the form $[H_{\text{min}}, E]$ where $H_{\text{min}}$ is the minimum value. It is only notationally more complicated to consider intervals $[E_1, E_2]$ with $E_1 > H_{\text{min}}, E_2 < H_{\text{max}}$.

**Theorem**
The density of states of the partial Bergman kernel is given by the asymptotic formulas:

$$
\Pi_{|kP}(z, z) \sim \begin{cases} 
  c_0 + c_1 k^{-1} + c_2 k^{-2} + \cdots, & \text{for } z \in H^{-1}(P), \\
  k^{-m} e^{-kb_E(z)} \left[ c_0(z) + O(k^{-1}) \right], & \text{for } z \in X_1^+, 
\end{cases}
$$

where $c_0 \in C^\infty(X_1^+)$, and $b_E$ is defined in (12). Furthermore, the remainder estimates are uniform on compact subsets of the basin $X_1^+$ of attraction of the minimum.
The smoothed out interval sums have the form, with $\rho \in S(\mathbb{R})$,

$$
\sum_j \rho\left(\frac{j}{k} - E\right) \Pi_{k,j}(z, z) = \int_{\mathbb{R}} \hat{\rho}(t) e^{-iE t} \Pi_k(e^{it/k} z, z) dt. \quad (13)
$$

**Theorem**

$$
\sum_j \rho\left(\frac{j}{k} - E\right) \Pi_{k,j}(z, z) \sim \begin{cases} 
  c_0 + c_1 k^{-1} + c_2 k^{-2} + \cdots, & z \in H^{-1}(P), \\
  k^{-m} e^{-kE(z)} \left[ c_0(z) + O(k^{-1}) \right], & z \in X_1^+, 
\end{cases}
$$

where the remainder estimates are uniform on compact subsets of the big stratum $X_1^+$ (big Morse cell).
The interface asymptotics at a level $E$ involve all of the individual weight Bergman kernels (8) where $|\frac{j}{k} - E| < \frac{C\log k}{k}$.

**Theorem**
For $z$ so that $\sqrt{k}(H(z) - \epsilon)$ is bounded, $k^{-n}\Pi_{|_{kP}}(z, z)$ has a distributional expansion on $X$ whose leading order term is

$$
k^{-n}\Pi_{|_{kP}}(z, z) = \frac{1}{\sqrt{2\pi|\xi_H(z)|^2}} \int_{-\infty}^{\sqrt{k}(H(z)-\epsilon)} e^{-\frac{t^2}{2|\xi_H(z)|^2}} dt + O(k^{-\frac{1}{2}}).
$$

$\xi_H = \text{Hamilton v.f. of } H.$
Smoothed interface asymptotics

The smoothed out interface sums have the form, with \( \rho \in S(\mathbb{R}) \),

\[
\sum_j \rho(\sqrt{k}(\frac{j}{k} - E)) \prod_{k,j}(z_0 + \frac{u}{\sqrt{k}}, z_0 + \frac{u}{\sqrt{k}})
\]

\[
= \int_{\mathbb{R}} \hat{\rho}(t)e^{-iEt\sqrt{k}} \prod_k(e^{it/\sqrt{k}}z, z)dt.
\]

**Theorem**

Let \( \mu(z_0) = E \). Then,

\[
\sum_j \rho(\sqrt{k}(\frac{j}{k} - E)) \prod_{k,j}(z_0 + \frac{u}{\sqrt{k}}, z_0 + \frac{u}{\sqrt{k}})
\]

\[
= \int_{\mathbb{R}} \hat{\rho}(t)e^{itE}\varphi'(z_0)u - \varphi''_{\rho\rho}(z_0)t^2 dt + O\left(\frac{1}{\sqrt{k}}\right).
\]
**Sketch of proof**

Using the Bouet-de-Monvel-Sjostrand parametrix,

$$
\sum_j f(\sqrt{k}(\frac{j}{k}-E))\Pi_k,j(z, z) = \int_\mathbb{R} \hat{f}(t)e^{-iE\sqrt{k}t}\Pi_k(e^{it}/\sqrt{k} z, z)dt
$$

$$
= k^m \int_{-\infty}^{\infty} \hat{f}(t)e^{-it(\sqrt{k}E)}e^{k\psi(e^{it/2\sqrt{k}} \cdot z, e^{-it/2\sqrt{k}} \cdot z)} - k\varphi(z) A_k(e^{it/2\sqrt{k}} z, z)\frac{dt}{2\pi}
$$

$$
+ k^m \int_{-\infty}^{\infty} \hat{f}(t)e^{-it(\sqrt{k}E)} R_k(e^{it/2\sqrt{k}} z, z)\frac{dt}{2\pi}
$$

Since $R_k \in k^{-\infty}C^\infty(M \times M)$, the second term is $O(k^{-\infty})$. We note that $t \rightarrow \Pi_k(e^{it}/\sqrt{k} z, z)$ is $2\pi\sqrt{k}$-periodic (similarly for the parametrix and remainder terms), so the integrals converge when $\hat{f} \in L^1(\mathbb{R}^d)$. 
Let
\[ \Psi(\tau, z) = -\tau(\sqrt{kE}) + k\psi(e^{\tau/2\sqrt{k}} \cdot z, e^{\bar{\tau}/2\sqrt{k}} \cdot z) - k\varphi(z), \]
so the phase is \( \Psi(it) \). If \( \varphi(z) \) is real analytic, then \( \Psi(\tau) \) is holomorphic when \( \Im(\tau) \) is small enough. If \( \varphi \) is only smooth, then \( \Psi(\tau) \) is an almost analytic extension of \( \Psi|_\mathbb{R} \).

If \( z = e^{\beta/(\sqrt{k})}z_0 \) with \( H(z_0) = E \). Then as \( k \to \infty \),
\[ \Psi(\tau, e^{\beta/(\sqrt{k})}z_0) = -\tau(\sqrt{kE}) + k((\psi(e^{(\tau/2+\beta)/\sqrt{k}} \cdot z_0, e^{(\bar{\tau}/2+\beta)/\sqrt{k}} \cdot z_0)
- \varphi(e^{\beta/\sqrt{k}} \cdot z_0)
= \frac{1}{2}((\tau/2 + \beta)^2 - \beta^2)\partial_\rho^2\varphi(z_0) + g_3(z, \tau, \beta), \]
where
\[ g_3 = O(k^{-1/2}(|\beta|^3 + |\tau|^3)). \]
Completion of proof

If $\hat{f} \in C_c(\mathbb{R})$, using the Plancherel theorem and the Taylor expansion above, the PBK is

$$k^m \int_{-\infty}^{\infty} \hat{f}(t) \left[ e^{\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial^2_{\rho}\varphi(z_0)} e^{g_3} dt \right] (1 + O(k^{-1}))$$

$$= k^m \int_{-\infty}^{\infty} \hat{f}(t) \left[ e^{\frac{1}{2}((it/2+\beta)^2-\beta^2)\partial^2_{\rho}\varphi(z_0)} dt \right] + O(k^{m-\frac{1}{2}}))$$

$$= k^m \int_{-\infty}^{\infty} f(x) \left[ \int_{-\infty}^{\infty} e^{-itx + \frac{1}{2}((it/2+\beta)^2-\beta^2)\partial^2_{\rho}\varphi(z_0)} dt \right] \frac{dx}{2\pi} + O(k^{m-\frac{1}{2}}))$$

$$= k^m \int_{-\infty}^{\infty} f(x) \sqrt{\frac{2}{\pi \partial^2_{\rho}\varphi(z_0)}} e^{-\frac{(2x-\beta\partial^2_{\rho}\varphi(z_0))^2}{2\partial^2_{\rho}\varphi(z_0)}} dx + O(k^{m-1/2}))$$